Optimization algorithms and the cosmological constant

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Outline

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Cosmological constant problem and the landscape

According to the Standard Model of particle physics,

- the energy density of the vacuum receives multiple contributions whose order of magnitude vastly exceeds the observed value $\Lambda \approx 1.5 \times 10^{-123} M_P^4$;[1]
- consistency with well-established cosmological history severely constrains large classes of approaches to this problem.[2]

In a landscape model,

- the universe can form large regions with different values of Λ;
- there are exponentially many ways of constructing a "vacuum;"
- observers necessarily find themselves in a highly atypical region that allows for a larger cosmological horizon.

Consistent with standard cosmological history if neighboring vacua have very different energies.[3]

[1] Perlmutter, et al., *Astrophys. J.*, 1999; Riess, et al., *Astro. J.*, 1998; Ade, et al., *Astron. Astrophys.*, 2016.

[2] Polchinski, hep-th/0603249; Bousso, *Gen. Rel. Grav.*, 2008.

[3] Bousso, Polchinski, *J. High Energy Physics*,, 2000.

Models of the landscape

Two simplified models of the landscape capture essential features:

- Arkani-Hamed-Dimopolous-Kachru (ADK) model[4], and
- Bousso-Polchinski (BP) model[5].

Here we focus on a simplification of the ADK model:

- the cosmological constant is obtained by summing the energy contributions from a large number of fields;
- each field is subject to a double-well potential;
- the two minima of each field to be a random number with mean zero and deviation of of order 1 in Planck units.

Given n such fields where vacuum energies $E_0^{(j)}$ and $E_1^{(j)}$, there are 2^n vacua, specified by $s \in \{0,1\}^n$:

$$\Lambda[s(j)] = \sum_{j=1}^{n} E_{s(j)}^{(j)}.$$

[4] Arkani-Hamed, Dimopolous, Kachru, hep-th/0501082

[5] Bousso, Polchinski, ibid.

Complexity

The ADK model is a variant of the number partitioning problem.

- This class of problems in NP-complete.

What cosmological dynamics solved the “hard” problem?

- The universe is exponentially expanding, creating new regions;
- gravity supplies resources for solving the problem;
- observers necessarily find themselves in the regions where a large problem has been solved.

Or, a local viewpoint trades the multiverse for “many worlds”

- one considers the different decay chains through the landscape;
- a patch decoheres rapidly when a vacuum transition takes place;
- observers find themselves in a branch that produced a vacuum with small Λ.

Computational censorship

Computational Censorship Hypothesis:

- a physical measurements should not access the solution to a problem that could not have been solved by the physical resources in the observable universe.

Possible definition of "resources" include:

- the Einstein-Hilbert-matter action,[6]
- the energy of the universe times its age[7];
- the maximum entropy of the visible universe;[8]
- the amount of entropy produced in our past light-cone.[9]

All given a number of gates $\Lambda^{-1} \approx 10^{122}$ (or slightly lower).

[6] Brown, Roberts, Susskind, Swingle, Zhao, *Phys. Rev. D*, 2016.

[7] Lloyd, *Phys. Rev. Lett.*, 2002

[8] Bousso, *JHEP*, 1999.

[9] Bousso, Harnik, Kribs, Perez, *Phys. Rev. D*, 2007.

Resolution

This leads to an "apparent paradox" in the ADK model.

- Resources available are $\approx \Lambda^{-1}$.
- Brute force search of the landscape scales as $\sim \Lambda^{-1} \left(\log_2 \Lambda^{-1}\right)^{3/2}$.

However this assumes n (number of fields in the ADK model) is such that Λ is an optimal solution to number partitioning.

For very large n, there are polynomial time (in n) heuristics.

1. There is no known way to bound how large n could be.
2. Karmarkar-Karp (specialized to number partitioning) can find "residues" of size Λ in time $\sim \exp \sqrt{\log \Lambda^{-1}}$.
3. Sieve algorithms (while exponential) are also very efficient and can be generalized past the ADK toy model.

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ADK reduction to number partitioning

The number partitioning problem is:

- given positive integers $\delta_1, \ldots, \delta_n$ to find $s_j \in \{+1, -1\}$ so that

$$\left| \sum_{j=1}^{n} s_j \delta_j \right| \leq 1.$$

Finding ADK vacua is very similar. Define

$$\delta_j = (E_1^{(j)} - E_0^{(j)})/2$$

Then

$$\Lambda = \sum_{j=1}^{n} s_j \delta_j.$$

So the numbers involved are real rather than integral.

Random instances of number partitioning

Random instances have been well studied using statistical mechanics.

- set some magnitude parameter B;
- sample n independent numbers $\delta_j \sim \text{Uniform}\{1, 2, \ldots, B\}$;
- define a perfect partition as $s_j = \pm 1$ so that

$$\sum_{j=1}^{n} s_j \delta_j = 0 \text{ if } \sum_{j=1}^{n} \delta_j \text{ even} \qquad \sum_{j=1}^{n} s_j \delta_j = 1 \text{ if } \sum_{j=1}^{n} \delta_j \text{ odd.}$$

Note for a random problem in the limit of large n,

- if $B > 2^{n+O(\log n)}$ will will likely be no perfect partitions,
- if $B < 2^{n+O(\log n)}$ there will be exponentially many partitions.

Note that if $B = \max_j \delta_j$ is only polynomially large, dynamic programming efficiently solves the number partitioning problem.

Cost of brute force search

Consider the number partitioning problem on real numbers.

- An instance is n numbers independently $\sim$ Uniform$[0,1]$.
- It is known that the median optimal residue is $\Theta(\sqrt{n}2^{-n})$.
- Thus, for a solution with residue Λ to exist, one needs $\sqrt{n}2^{-n} < \Lambda$.

This gives problems with

$$n \sim \log_2 \Lambda^{-1} + \frac{1}{2} \log_2 \log_2 \Lambda^{-1} \text{ with}$$
$$b \sim \log_2 \Lambda^{-1} \text{ bits of precision.}$$

Naively, the total complexity of enumeration is $O(nb2^n)$. Instead:

- order tuples $(s_j)_{j=1}^n$ according to a binary reflective Gray code;
- consecutive tuples only differ in one index;
- use the residue from the previous step and add or subtract $2\delta_j$;
- the total complexity of the algorithm is $O(b2^n)$ elementary gates.

This yields a total complexity of order $\Lambda^{-1} \left(\log_2 \Lambda^{-1}\right)^{3/2}$.

Outline

Karmarkar-Karp

The Karmarkar-Karp heuristic is that the largest numbers should be given opposite sign to maximize (relative) cancellation.[10]

- extract the two largest numbers from the list;
- compute their (positive) difference;
- insert the difference back into the list of numbers.

This reduces the problem to a new instance of number partitioning with one fewer number.

The work to perform Karmarkar-Karp goes as

- sorting the initial list has complexity $O(n \log n)$;
- using a heap allows inserting numbers with complexity $O(\log n)$;
- there as exactly $n - 1$ differencing-and-insertion steps.

Thus the total complexity of the algorithm is $O(n \log n)$.

[10] Karmarkar, Karp, *FOCS*, 1982.

Karmarkar-Karp

to find the cosmological constant

Note that if $n \gg \log_2 \Lambda^{-1}$ then $\Lambda^{-1} \ll b2^n$.

- Even the best known exact algorithm scales as $\sim 2^{0.241n}$.

Karmarkar-Karp scales as $n \log(n)$, however will not generally find a perfect partition.

- The K-K residue has smaller size by a factor of $\approx e^{-c \log^2 n}$.
- Asymptotically as $n \to \infty$, we have $c = \frac{1}{\sqrt{2}}$.[11]

For the ADK model, for K-K to find a residue of size Λ

- $n \sim \exp\left[\sqrt{\frac{\log \Lambda^{-1}}{c}}\right]$,

[11] Yakir, *Math. Operations Research*, 1996

Exact algorithms

Number partitioning, subset sum, and knapsack problems are basically the same.

Exact algorithm exists that run in $O(2^{\alpha n})$.

- A straightforward meet-in-the-middle tree search is $\approx 2^{0.5n}$.[12]
- The best known classical algorithm takes $O(2^{0.291n})$.[13]
- The best known quantum algorithm takes $O(2^{0.241n})$.[14]
- The adiabatic algorithm has unknown runtime, but appears to scale as $2^{0.8n}$.[15]

We can use exact algorithms "locally" to produce a sieve heuristic.

- We need to understand the statistics of optimal solutions.

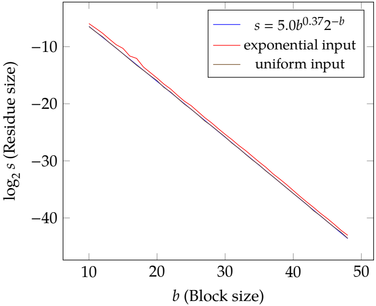
[12] Horowitz, Sahni, *Journ. ACM*, 1974.

[13] Becker, Coron and Joux, *EUROCRYPT*, 2011.

[14] Bernstein, Jeffery, Lange, Meurer, *Post-Quantum Cryptography*, 2013.

[15] Johnson, Aragon, McGeoch, Schevron, *Operations Research*, 1991.

Statistics of the optimal residue (mean)



The optimal residue scales as $\Theta(\sqrt{b}2^b)$.

- Ordinate = mean relative optimal residue size.
- Abscissa = input size for number partitioning problem.
- 1000 instances solved for each $b \in \{10, \ldots, 48\}$.
- Computed with least square error estimator (exponential model).

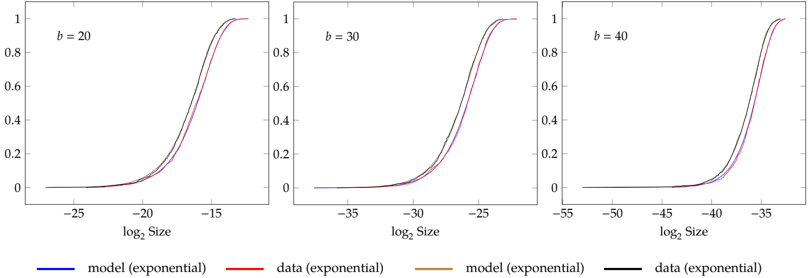
The model $s = 5.0b^{0.37}2^{-b}$ was generated by linear regression.

Statistics of the optimal residue (distribution)

Distribution of the optimal residue for NPP sizes $b = 20, 30, 40$.

- Ordinate = cumulative probability distribution.
- Abscissa = $\log_2$ optimal residue size.
- 1000 data points per plot.

The model is the exponential distribution with mean computed by least squares estimator on data.



A sieve for number partitioning

Here we explore a very simple sieve mechanism.

- ”Lattice sieves” add lattice vectors to produce smaller vectors.
- The simple sieve here is similar in spirit to “tuple” sieve.[16]

In general, a sieve consists of several stages.

- Partition in the input in to small problems.
- Use an exact algorithm to solve the subproblem.
- The results form the input for the next stage of the sieve.

For number partitioning problems,

- we partition all the numbers into blocks of size b,
- we use one the exact methods above, taking work $2^{\alpha b+o(b)}$.

Residues are exponentially distributed with expected size $2^{-b+o(b)}$.

[16] Bai, Laarhoven, Stehlé, *ANTS*, 2016.

A sieve for number partitioning

First stage of the sieve:

Input: n fields of mean energy differences $\delta \approx 1$.

Solve: n/b_1 number partition problems, each of size b_1.

Output: $\frac{n}{b_1}$ residues with mean size $\approx 2^{-b_1}$.

Work: $\approx \frac{n}{b_1} 2^{\alpha b_1}$.

Second stage of the sieve:

Input: n/b_1 residues with mean size $\approx 2^{-b_1}$.

Solve: $n/(b_1 b_2)$ number partition problems, each of size b_2.

Output: $\frac{n}{b_1 b_2}$ residues with mean size $\approx 2^{-(b_1+b_2)}$.

Work: $\approx \frac{n}{b_1 b_2} 2^{\alpha b_2}$.

And so on. After k stages:

Output: single residue of expected length $2^{-t} \approx 2^{-(b_1+\cdots+b_k)}$

Work: $\approx \left(\frac{n}{b_1} 2^{\alpha b_1} + \frac{n}{b_1 b_2} 2^{\alpha b_2} + \cdots + \frac{n}{b_1 b_2 \cdots b_k} 2^{\alpha b_k} \right)$.

A sieve for number partitioning

The optimal work is given by an "equipartition principle:"

- balance the amount of work done on each sieve stage.

Specifically, for stage j and $j-1$ we want

$$\frac{n}{b_1 \cdots b_j} 2^{\alpha b_j} \approx \frac{n}{b_1 \cdots b_{j-1}} 2^{\alpha b_{j-1}} \text{ so we take } b_j - \tfrac{1}{\alpha} \log_2(b_j) \approx b_{j-1}.$$

This table was generated with $\alpha = 0.5$:

k	n	t	$\log_2(\text{Work})$	$(b_1, b_2, \dots)$
2	4.22×10^4	400.0	107.62	(198, 213)
3	2.65×10^6	400.8	78.32	(124, 139, 154)
4	1.19×10^8	400.8	65.07	(85, 98, 113, 126)
5	3.96×10^9	400.0	58.14	(59, 72, 85, 98, 112)
6	1.03×10^{11}	400.3	54.53	(41, 53, 65, 77, 91, 104)
7	1.97×10^{12}	400.8	52.70	(27, 38, 49, 61, 74, 87, 100)
8	2.54×10^{13}	400.5	51.88	(16, 26, 36, 48, 59, 72, 85, 98)

(Karmarkar-Karp takes $n \approx 7.8 \times 10^8$ and runs in $w \approx 36.5$.)

Outline

Experiment: Karmarkar-Karp

To empirically test the Karmarkar-Karp algorithm in a regime relevant to the cosmological constant problem within the ADK model:

- one starts with real numbers of order 1, and seeks to find a residue of order $\sim 10^{-120} \approx 2^{400}$;
- scale by a factor of 2^{430} to represented these as integers;
- defined success as achieving residue less than 2^{30}.

The 30 bits of precision deal with "numerical noise."

Therefore an experiment was:

- take n independent $\sim \text{Uniform}\{0, 1, 2, \ldots, 2^{430} - 1\}$,
- test if Karmarkar-Karp achieves a residue less than 2^{30}.

We performed 200 trials for a variety of n around the prediced threshold.

Experiment: Karmarkar-Karp

Predictions

Theory predicts the size of the final residue as exponentially distributed:

- $\Pr\{y < Y < y + dy\} = \lambda e^{-\lambda y} dy$,
- the key parameter is modeled as $\lambda = e^{-c \log^2 n}$,
- asymptotically $c \to 1/\sqrt{2}$ (smaller c are consistently observed).

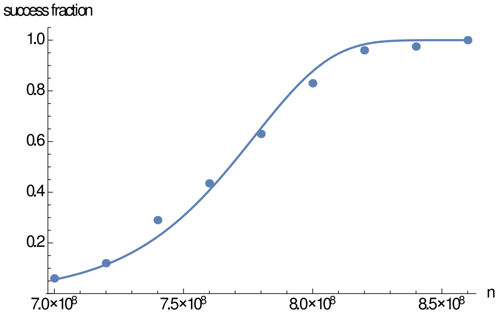
For a reduction factor of $\epsilon = 2^{-400}$, we obtain success probability

$$
\begin{aligned}
P &= \int_0^{\epsilon} \lambda e^{-\lambda y} dy \\
&= 1 - \exp\left[-e^{-c \log^2 n} \epsilon\right].
\end{aligned}
$$

We can use this to fit c to empirical data.

Experiment: Karmarkar-Karp

Numerical results



Plots for a Karmarkar-Karp experiment.

- Ordinate = likelihood in 200 trials of a residue $< 2^{30}$.
- Abscissa = number of samples $\sim$ Uniform$\{0, \ldots, 2^{430} - 1\}$.
- Theory curve $= 1 - \exp\left(-e^{c \log(n)^2} / 2^{400}\right)$.
- Parameter $c = 0.6615$ is fitted. (Asymptotic prediction: $\frac{1}{\sqrt{2}}$.)

Experiment: a toy NPP sieve

As a simple proof of concept, we tackle a toy sieve:

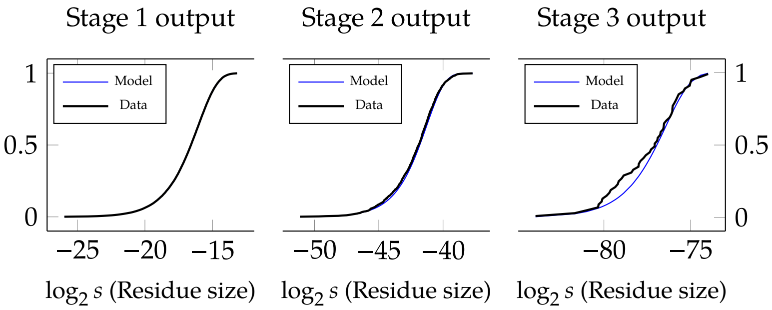
- four stages;
- block sizes $(b_1, b_2, b_3, b_4) = (20, 30, 40, 50)$;
- a simple meet-in-the-middle algorithm ($\alpha = 0.5$).

Stage	b	#Inputs	Distribution	#NPPs	Work	$\mathbb{E}[s]$
One	20	1200000	Uniform	60000	$2^{25.9}$	$2^{-16.1}$
Two	30	60000	Exponential	2000	$2^{26.0}$	$2^{-41.3}$
Three	40	2000	Exponential	50	$2^{25.6}$	$2^{-76.4}$
Four	50	50	Exponential	1	$2^{25.0}$	$2^{-121.3}$

- Work quote is for the entire sieve stage.
- Expected residue size is based on distributional model.

Experiment: a toy NPP sieve

Numerical results



Plots for a four state sieve.

- Ordinate = cumulative likelihood of observing the value.
- Abscissa = (log) size of the optimal residue obtained.
- This final residue was $6.54 \times 10^{-38} \lesssim 2^{-121.3}$.
- The sieve took 152 seconds on my Mac Pro.