Distribution of Graph States via Quantum Routers with Network Coding

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INTRODUCTION

Quantum repeaters are devices that subdivide a long quantum channel into smaller segments [1, 2]. They are designed to tackle errors like noise and losses which would spoil the transmitted quantum state. Long-distance quantum communication via quantum repeaters allows distant parties to perform various quantum information protocols. A prominent application is quantum key distribution [3–5].

Several proposals for quantum repeaters are known. They may be classified by the direction of classical communication: two-way communication is used for repeat-until-success strategies for transmission between neighbouring repeaters [6–8] and for entanglement distillation protocols, while one-way communication suffices for schemes based on quantum error correction [9–12]. Our results can be obtained for both types of repeaters, but we focus on repeaters of the latter kind in this talk [13].

The previous descriptions apply to a single channel connecting two parties A and B. It is natural to consider the multipartite generalization of this scenario: several sites A, B, C, ... are connected by quantum channels. Such a quantum network corresponds to a mathematical graph, where the sites are represented by nodes and the quantum channels are the edges of the graph. This way the mathematical graph models the physical infrastructure of the quantum network. In practice the capacity of each link is constrained. For simplicity we assume that each channel allows to transmit a single qudit of fixed dimension D per time step. Some sites are special in the sense that they will share a qudit of the finally distributed entangled quantum state. We continue to call these sites parties, while we call the other nodes quantum repeaters or quantum routers if they have vertex degree two or larger than two, respectively.

RESEARCH QUESTION

The task that we consider is the distribution of entangled states (more precisely graph states, see below) shared by arbitrary subsets of parties via a quantum network. These states are a general resource for different quantum information protocols, e.g. teleportation [14], quantum key distribution [3–5], distributed quantum computing [15], secret sharing [16] and Bell test experiments [17].

We investigate the performance of such an entanglement distribution protocol with respect to

1. the overall throughput of the quantum network, i.e. the production rate for the desired quantum state,
2. and the robustness against failures of nodes (e.g. due to a power outage).

The same questions arise in classical communication networks [18, 19]. It turned out that network coding, i.e. routers that can modify the transmitted data, can be advantageous with respect to the throughput and the robustness of the network, compared to routing-only strategies. We investigate similar advantages in the quantum scenario described above. Additionally we generalize a known link between classical linear network codes and quantum network codes [20] to the distribution of quantum graph states.

SKETCH OF THE PROTOCOL

The quantum network coding protocol we investigate can be summarized as follows. The quantum graph state associated with the network described above is distributed using the graph state repeater scheme of [13]. Transmission losses are corrected using a quantum error correction code [21] following that reference (this physical layer will be hidden from the abstraction in this talk). Then all repeaters and all routers measure their qudit in X-basis, i.e. the repeaters and routers perform a special type of measurement-based quantum computation. By-product operators depending on the measurement outcomes are applied, such that the protocol deterministically produces the same quantum state for all outcomes. Note that these measurements can be done on-the-fly, such that the large graph state is never actually present. Nevertheless it gives a very convenient description of the quantum network.

METHODS

We use the stabilizer formalism. Instead of writing down the quantum state $|\psi\rangle$ of a system it can be more convenient to keep track of a set S of commuting operators, which uniquely define $|\psi\rangle$ via the eigenequations

$$s|\psi\rangle = |\psi\rangle \qquad \forall s \in S. \tag{1}$$

We say s stabilizes $|\psi\rangle$. Please note that products of stabilizer operators fulfil Eq. (1), too.

Consider a measurement $\mathcal{A}$ with outcome a. One may update S corresponding to the new knowledge by replacing all $s \in S$

that commute with the projector P_a on the eigenspace of $\mathcal{A}$ associated with the eigenvalue a by $P_a s$. The other stabilizer operators are invalidated. Let the post-measurement state $|\psi'\rangle$ be defined by a minimal set of operators that commute with P_a (for all a). We call these operators the main stabilizers. Identifying them reveals the measurement dynamics.
We will be mostly interested in the non-measured part B of a system composed of A and B after measuring $\mathcal{A}$ on A with outcome a. If s has the form $\mathcal{A}^x \otimes \mathcal{B}$, then we simply replace it by $a^x\mathcal{B}$.

The stabilizer formalism is particularly handy in the context of graph states [22, 23]. The D-dimensional graph state $|G\rangle$ associated with the mathematical graph $G = (V, E)$ with vertex set V, edges E, and adjacency matrix Γ is stabilized by the so-called stabilizer generators

$$g_v = X_v \prod_{w \in V} Z_w^{\Gamma_{vw}} \qquad \forall v \in V, \tag{2}$$

$$\text{where } X = \sum_{j=0}^{D-1} |j+1 \bmod D\rangle\langle j| \tag{3}$$

$$\text{and } Z = \sum_{j=0}^{D-1} e^{j2\pi i/D} |j\rangle\langle j|. \tag{4}$$

X and Z are not Hermitian but normal and we will use them as observables. The discrete Fourier transform matrix

$$H = \frac{1}{\sqrt{D}} \sum_{xy} e^{xy\frac{2\pi i}{D}} |x\rangle\langle y| \tag{5}$$

performs the basis change $HXH^\dagger = Z$ and $HZH^\dagger = X^{-1}$.

The concept of X-chains, i.e. stabilizer operators that contain only X-operators (products of stabilizer generators where all Z contributions cancel out), is very useful when analysing the robustness of the quantum network coding scheme with respect to node failures. The main idea is that the outcomes of the X-measurements on the repeaters and routers allow to determine error syndromes for X-chains, because the product of the measurement outcomes would be 1 in the ideal case. Note that it is not possible to directly measure a stabilizer generator in our scenario, because the qudits are located at different sites of the network.

INSTRUCTIVE EXAMPLES

The most prominent example of network coding is the so-called butterfly network [24, 25], see Figure 1. The desired target state, a tensor product of two Bell pairs, is also shown in Figure 1. One can easily see that this task is achieved by measuring the routers 1 and 2 in X-basis (and applying by-product operators to correct for the measurement outcomes) by looking at the following main stabilizers: $X_{A_1}X_2Z_{B_1}$, $X_{A_2}X_2Z_{B_2}$, $X_{B_1}X_1Z_{A_1}$ and $X_{B_2}X_1Z_{A_2}$. The measurements project these main stabilizers onto the stabilizer generators of the target state.

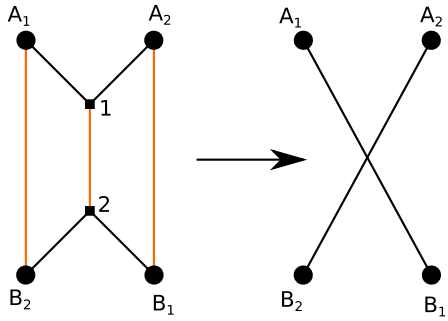
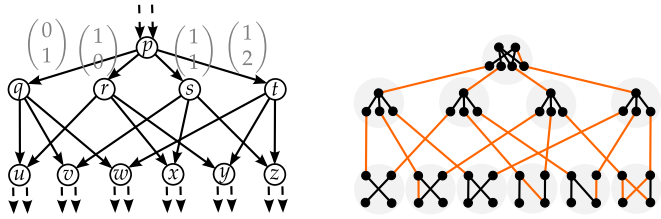


FIG. 1. The butterfly network. A_1 and A_2 want to share a Bell pair with B_1 and B_2, respectively. Orange edges have weight -1 and edge weights of 1 are omitted.



(a)A ternary linear network coding.

(b)The corresponding quantum network coding.

FIG. 2. A linear network code (taken from [18], Fig. 19.5) and its corresponding quantum network code (QNC). The network does not allow for a binary linear network code to achieve the task. The QNC produces two 7-qutrit GHZ states. Orange edges have weight -1 and edge weights of 1 are omitted.

Figure 2 shows a quantum network code that distributes two seven-qutrit GHZ states (star-graph states) by measuring the qutrits from the second row to the second-last row in X-basis. This example is based on Fig. 19.5 of []. It illustrates the distribution of multipartite entangled states. Again the functionality of the network coding can be understood by finding appropriate main stabilizers that connect the upper party with the lower party via chains of X-operators.
A network code variant of the nine-qudit-Shor code is shown in Figure 3. This network produces a Bell pair (up to by-product operators) shared by parties a and b if all other qudits are measured in X-basis (one can easily check this from the main stabilizers). Furthermore this network can tolerate the

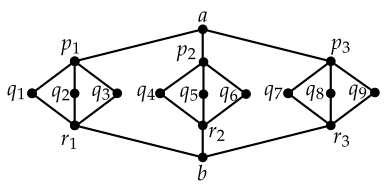


FIG. 3. A network variant of the nine-qudit-Shor error correction code [21].

failure of one of the nodes q_1, q_2, ..., q_9. One way to see this is that it is always possible to find main stabilizer operators that act trivially on the lost qudit. Or, equivalently, one can use the syndrome measurement of the X-chains (e.g. $g_{q_1}g_{q_2}^{-1}$, $g_{p_1}g_{p_2}^{-1}g_{r_1}g_{r_2}^{-1}$) as described above to correct the error.

SUMMARY OF MAIN RESULTS

The main results presented in this talk are the following. First we generalize the result of [20] to the distribution of arbitrary graph states associated with bipartite graphs, i.e. a graph that allows a split into two partitions, s.t. there are no edges between two vertices of the same partition. The quantum network code allows to distribute this state in a single use of the network. There are examples where this is not possible with a routing strategy, i.e. one observes a gain of throughput in comparison to the scenario where the routers are not able to perform computations on the transmitted qudits.
Second we show how X-chains in the qudit graph state associated with the network graph give error syndromes. This leads to network based error correction in graph state repeater networks. The analysis of the robustness of the network coding against node failures is greatly simplified by mapping network codes as described above to stabilizer error correction codes. An error in the network can be corrected if and only if a corresponding error can be corrected in the stabilizer error correction code. Furthermore we give the reverse direction, i.e. we describe an explicit construction of a network code derived from a stabilizer error correction code (see also Figure 3).
Third we give simple rules to simplify a quantum network code of the type discussed above. This is done by investigating which networks and their associated quantum graph state lead to the same final state. Basically it is achieved by checking whether the post-measurement state of the X-measurement on a certain qudit of the graph state leads to a graph state that is compatible with the network constraints (the rate constraints of the edges), i.e. whether the post-measurement state could have been produced without the "detour".

OUTLOOK

The formalism developed in this work can be applied to investigate further interesting questions in this context. We plan to apply the techniques developed to simplify the quantum network code to the simplification of gates in measurement-based quantum computation, e.g. the SWAP gate that corresponds to the butterfly network.
It will be interesting to consider network coding in random graphs and compare it to classical random network coding, which proved to be very efficient, e.g. if the network layout is unknown.
A further related project is the generalization of the presented scheme to the distribution of graph states associated with non-bipartite graphs.
The advantage of the distribution of multipartite entanglement via routers with network coding for quantum key distribution, as measured by the secret key rate, compared to bipartite QKD protocols, deserves further quantitative investigations.

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