Quantum error-correcting codes for a bosonic mode

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(Dated: April 29, 2016)

Successful transmission of quantum information over long distances is a cornerstone of quantum cryptographic protocols and remains a daunting experimental challenge. Photons remain the medium of choice for facilitating such transmissions, and the community has typically focused on transmitting information in only a small number of "flying" photons. Common examples include encoding a qubit in two orthogonal polarizations of a single photon or encoding two qubits in a pair of photons entangled in energy and time [1]. If any such photons are lost during flight, the corresponding encoded information is unrecoverable. However, the large (i.e., infinite) Hilbert space of a photonic mode offers the possibility of utilizing encodings which allow for recovery of the information despite photon loss (or other errors) occurring mid-flight. Needless to say, such encodings are also useful for storing and protecting quantum information in stationary photonic media (e.g., microwave cavities [2]).

In contrast to qubit-based schemes, error correction for the infinite bosonic Hilbert space in principle requires the consideration of an infinite number of error operators. For example, during a finite time interval in a lossy system, there is a finite probability of an arbitrary number of photons being lost, completely annihilating any finite code state. Similar to quantum error-correcting codes correcting only the single-qubit errors of a channel containing both single- and multi-qubit errors, photonic error-correction techniques are "approximate" [3] in the sense that they will only be able to correct against a subset of all errors of a physical channel. Furthermore, the action of physical error operators such as the photon loss operator is strongly correlated across different Fock states (i.e., photon number eigenstates), making the straightforward transfer of multi-qubit schemes to a single bosonic mode impossible.

Two classes of single-mode codes have previously been proposed to achieve recoverability: the seminal Gottesman-Kitaev-Preskill (GKP) codes [4, 5], constructed to protect from small shifts in photonic quadratures, and cat-codes [6, 7], consisting of superpositions of evenly distributed coherent states. Code states of both classes consist of superpositions of an infinite number of Fock states, making encoding arguably more complex as compared to code states defined on a finite subspace.

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Here, we propose a new class of bosonic codes, *the binomial codes* [8]. The binomial code states are formed from a finite superposition of Fock states weighted with square roots of binomial coefficients. The codes can exactly correct errors that are polynomial up to a specified degree in photonic creation and annihilation operators, including amplitude damping and displacement noise as well as photon addition and dephasing errors. Besides being conceptually simple and highly customizable, binomial codes can protect quantum information from certain errors using a smaller average photon number than the corresponding cat codes. The binomial codes are tailored for detecting photon loss and gain errors by means of measurements of the generalized photon number parity, which is favorable for implementation in state-of-the-art experimental schemes [9]. In Ref. [8], we present an explicit quantum error recovery operation based on projective measurements and unitary operations.

Additionally, we relax the aforementioned generalized parity structure of the binomial codes and numerically obtain codes with even lower unrecoverable error rates and smaller average photon number. Interestingly, some of these *numerically optimized* photonic codes can be expressed in closed form.

Below we describe the main features of both sets of codes, emphasizing the binomial codes' customizability for correcting arbitrary combinations of photonic creation and annihilation operators. Both the binomial and numerically optimized codes should prove useful in several quantum technologies, including photonic quantum communication, optical-to-microwave up- and down-conversion, and bosonic quantum memories.

NEW CLASSES OF PHOTONIC CODES

Suppose that flying quantum information is subjected to a error/noise channel $\mathcal{E}_\gamma$ that can be expanded in a small parameter $\gamma \ll 1$. The goal of quantum error correction is to find an encoding (denoted by projection P) and a recovery operation $\mathcal{R}$ such that the effect of the error is suppressed to some higher order L after application of the recovery:

$$\rho = P\rho P \longrightarrow \mathcal{R}\mathcal{E}_\gamma(\rho) = \rho + O\left(\gamma^{L+1}\right) . \qquad (1)$$

For many physical error channels acting on multi-qubit systems, the γ-expansion of the error channel's Kraus operators consists of sums of products of single-qubit Pauli operators whose weight increases with the order in γ [10]. If the first few terms in the expansion take the code states to distinct subspaces of orthogonal error states, then those terms are correctable and the corresponding order in γ is suppressed after recovery. Quantitatively, this is represented by the Knill-Laflamme quantum error correction conditions [11]. For example, a pair of elements $\{E_1, E_2\}$ in the expansion of $\mathcal{E}_\gamma$ is correctable if and only if

$$PE_k^\dagger E_\ell P = c_{k\ell} P \tag{2}$$

for $\ell, k \in \{1, 2\}$. If the above is satisfied, then there exist syndromes which allow one to detect and correct the two corresponding errors during the recovery operation $\mathcal{R}$.

While a single mode does not consist of multiple physical qubits, we develop a similarly useful expansion in terms of the raising ($\hat{a}^\dagger$) and lowering ($\hat{a}$) operators of the mode (with $\left[\hat{a}, \hat{a}^\dagger\right] = 1$). Analogous to a multi-qubit code which protects from all single-qubit errors (i.e., operators of weight 1), there exists a binomial code which protects from single powers of $\hat{a}$ and $\hat{a}^\dagger$. We can also carry over the principle of superposition that is so prominent in multi-qubit error correction. Just as a multi-qubit code protecting against Pauli X and Z single-qubit errors *also* protects against arbitrary single-qubit errors, a binomial code protecting against operators $\hat{a}^{\dagger k}\hat{a}^\ell$ for $k, \ell \leq L$ *also* protects against any operators which can be written as a superposition of $\hat{a}^{\dagger k}\hat{a}^\ell$.

Binomial codes: simple example

A simple example of the above framework is the smallest binomial code

$$|W_\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |4\rangle\right) \qquad \text{and} \qquad |W_\downarrow\rangle = |2\rangle\,, \tag{3}$$

where $|n\rangle$ with $n \geq 0$ are the photonic Fock states. This code protects either against the pair $\{I, \hat{a}\}$ or $\{I, \hat{a}^\dagger\}$, where I is the identity (i.e., no error). One readily observes that the codes consist of Fock states of even photon numbers. This *spacing* guarantees that, upon loss (or gain) of a photon, the resulting error states remain orthogonal to the code space. Upon action of $\hat{a}$ on the code states, the resulting states $\hat{a}|W_\uparrow\rangle \propto |3\rangle$ and $\hat{a}|W_\downarrow\rangle \propto |1\rangle$ are located in the odd-photon-number subspace and are thus orthogonal to the even-subspace code words. In addition, the two error states are spaced far enough to be orthogonal to each other. The corresponding syndrome used to detect a photon loss (or gain) event is simply the photon number parity $(-)^{\hat{a}^\dagger\hat{a}}$.

However, since the code space projection is $P = |W_\uparrow\rangle\langle W_\uparrow| + |W_\downarrow\rangle\langle W_\downarrow|$, quantum error correction conditions (2) also require that $\langle W_\uparrow|\hat{a}^\dagger\hat{a}|W_\uparrow\rangle = \langle W_\downarrow|\hat{a}^\dagger\hat{a}|W_\downarrow\rangle$. This condition is equivalent to the code words having the same average photon number, which can be verified by direct observation of Eq. (3). We will show this in a different way to demonstrate why the codes are named as such. Superimposing the code words yields

$$|W_\pm\rangle = \frac{1}{2}\left(|0\rangle \pm \sqrt{2}|2\rangle + |4\rangle\right)\,, \tag{4}$$

where the coefficients are square roots of the binomial coefficients "1 2 1" from the third line of Pascal's triangle. Note that in this basis, the quantum error correction conditions (2) can be proven using the binomial formula:

$$\langle W_+|\hat{a}^\dagger\hat{a}|W_\pm\rangle = \frac{1}{2}\sum_{n=0}^{2}\binom{2}{n} n\,(\pm)^n = \frac{x}{2}\frac{d}{dx}\left(1 \pm x\right)^2\bigg|_{x=1}.$$

Binomial codes: general case

The family of binomial codes is expressed as

$$|W_{\uparrow/\downarrow}^{N,S}\rangle = \frac{1}{\sqrt{2^N}}\sum_{p \text{ even/odd}}^{[0,N+1]}\sqrt{\binom{N+1}{p}}|p(S+1)\rangle\,, \tag{5}$$

with spacing $S > 0$, order $N > 0$, and p ranging from 0 to $N+1$. The example from the previous Subsection is the $N, S = 1$ case. The previous analysis and use of the binomial formula can be straightforwardly extended to show that a code space spanned by the two codewords satisfies the quantum error correction conditions (2) for all $\hat{a}^{\dagger k}\hat{a}^\ell$ such that $|k - \ell| \leq S$ and $k + \ell \leq 2N$. This means that any elements of the small γ expansion of the error channel $\mathcal{E}_\gamma$ which consist of a linear superposition of such $\hat{a}^{\dagger k}\hat{a}^\ell$ can be corrected. Therefore, codes at different points of the two-dimensional parameter space $\{N, S\}$ are tailored to protecting against different types of errors. Codes with $S \gg N$ protect against error channels which cause large photon losses while codes with $S = 1 \ll N$ protect against "dephasing" error channels expressible in powers of $\hat{a}^\dagger\hat{a}$.

As a real-world example, we can consider the photonic amplitude damping channel whose Kraus operators are $E_\ell = \sqrt{\frac{(1-e^{-\gamma})^\ell}{\ell!}}\, e^{-\frac{1}{2}\gamma\hat{a}^\dagger\hat{a}}\hat{a}^\ell$. For an optical fiber, the damping factor $\gamma = l/l_{\text{att}}$ with l being the length of the channel and l_{att} being the attenuation length. For a stationary cavity, $\gamma = \kappa\delta t$ with δt being time and κ being the photon loss rate. The Kraus operators in the order-L expansion in γ for such a channel are of the form $\hat{a}^{\dagger k}\hat{a}^\ell$ with $k, \ell \leq L$. Therefore, setting $L = S = N$ allows one to satisfy Eqs. (1-2) and recover the information to the desired order.

Numerically optimized codes

The spacing between binomial code words which provides correction against photon losses comes at a price — an average photon number increasing linearly with S. We have used several numerical schemes which utilize the quantum error conditions (2) for the first few powers of $\hat{a}$ and obtained codes which do not have a spacing, have a smaller average photon number, and still correct against the chosen errors to the desired order. Surprisingly, some of these codes can be obtained analytically. For example, the code

$$\begin{aligned} |W_\uparrow\rangle &= \tfrac{1}{\sqrt{6}}\left(\sqrt{7-\sqrt{17}}|0\rangle + \sqrt{\sqrt{17}-1}|3\rangle\right) \\ |W_\downarrow\rangle &= \tfrac{1}{\sqrt{6}}\left(\sqrt{9-\sqrt{17}}|2\rangle - \sqrt{\sqrt{17}-3}|4\rangle\right) \end{aligned} \tag{6}$$

has an average photon number of approximately 1.56, compared to 2 for the smallest binomial code (3). A careful calculation ([8], Appx. H) reveals that this code is capable of correcting errors to first order in the γ-expansion of the amplitude damping channel.

OUTLOOK

With the advent of binomial and numerically optimized codes in addition to the existing GKP and cat code families, there are currently (at least) four families of single-mode encodings. This raises the question: *Which encoding is best?* Expanding in the small parameters of the channel may not be sufficient to answer this question since there are many other degrees of freedom not taken into account. These include the average photon number, the employed recovery channel $\mathcal{R}$, fidelity metric, and overall experimental feasibility. In the case of GKP codes, another obstacle is the error model: those codes have not yet been thorough analyzed in terms of the photon loss and creation operators $\hat{a}$ and $\hat{a}^\dagger$. An implementation-independent appraisal of the various codes could begin by making use of channel-adapted quantum error recovery [12, 13]. A comparison of the best case recovery fidelities for the various codes should prove helpful in determining code applicability to various error channels.

ACKNOWLEDGMENTS

We are grateful for useful discussions with Huaixiu Zheng, Reinier W. Heeres, Philip Reinhold, Hendrik Meier, Linshu Li, John Preskill, N. Read, Konrad W. Lehnert, Mazyar Mirrahimi, Barbara M. Terhal, Michel H. Devoret and Robert J. Schoelkopf. We acknowledge support from ARL-CDQI, ARO W911NF-14-1-0011, W911NF- 14-1-0563, NSF DMR-1301798, DGE-1122492, AFOSR MURI FA9550-14-1-0052, FA9550-14-1-0015, Alfred P. Sloan Foundation BR2013-049, and the Packard Foundation 2013-39273.

[1] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[2] N. J. Cerf, G. Leuchs, and E. S. Polzik, *Quantum Information with Continuous Variables of Atoms and Light* (World Scientific, London, 2007).
[3] D. W. Leung, M. A. Nielsen, I. L. Chuang, and Y. Yamamoto, Phys. Rev. A **56**, 2567 (1997).
[4] D. Gottesman, A. Yu. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).
[5] D. Gottesman and J. Preskill, Phys. Rev. A **63**, 022309 (2001).
[6] Z. Leghtas, G. Kirchmair, B. Vlastakis, R. J. Schoelkopf, M. H. Devoret, and M. Mirrahimi, Phys. Rev. Lett. **111**, 120501 (2013).
[7] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret, New J. Phys. **16**, 045014 (2014).
[8] M. H. Michael, M. Silveri, R. T. Brierley, V. V. Albert, J. Salmilehto, L. Jiang, and S. M. Girvin, arXiv:1602.00008.
[9] N. Ofek, A. Petrenko, R. Heeres, P. Reinhold, Z. Leghtas, B. Vlastakis, Y. Liu, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, arXiv:1602.04768.
[10] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2011).
[11] E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[12] A. S. Fletcher, *Channel-Adapted Quantum Error Correction*, Ph.D. thesis (2007), arXiv:0706.3400.
[13] A. S. Fletcher, P. W. Shor, and M. Z. Win, Phys. Rev. A **75**, 012338 (2007).