nature physics

Article

https://doi.org/10.1038/s41567-024-02621-x

Complexity and order in approximate quantum error-correcting codes

Received: 20 December 2023

Accepted: 24 July 2024

Published online: 03 September 2024

Check for updates

Jinmin Yi[1,2], Weicheng Ye[1], Daniel Gottesman[3] & Zi-Wen Liu[4]

Some form of quantum error correction is necessary to produce large-scale fault-tolerant quantum computers and finds broad relevance in physics. Most studies customarily assume exact correction. However, codes that may only enable approximate quantum error correction (AQEC) could be useful and intrinsically important in many practical and physical contexts. Here we establish rigorous connections between quantum circuit complexity and AQEC capability. Our analysis covers systems with both all-to-all connectivity and geometric scenarios like lattice systems. To this end, we introduce a type of code parameter that we call subsystem variance, which is closely related to the optimal AQEC precision. For a code encoding k logical qubits in n physical qubits, we find that if the subsystem variance is below an $O(k/n)$ threshold, then any state in the code subspace must obey certain circuit complexity lower bounds, which identify non-trivial phases of codes. This theory of AQEC provides a versatile framework for understanding quantum complexity and order in many-body quantum systems, generating new insights for wide-ranging important physical scenarios such as topological order and critical quantum systems. Our results suggest that $O(1/n)$ represents a common, physically profound scaling threshold of subsystem variance for features associated with non-trivial quantum order.

A pillar of quantum information science and technology quantum error correction (QEC) has been extensively studied as a means to protect quantum information from noise and errors for the purpose of realizing the potential advantages of quantum computation in practice[1–3]. Remarkably, in recent years, it has become increasingly evident that the concept of QEC carries broad importance in fundamental physics, extending far beyond its original realm. In particular, QEC plays fundamental roles in our understanding of topological order[4] and anti-de Sitter (AdS)/conformal field theory (CFT) correspondence[5] that stand at the frontier of many-body physics and quantum gravity, respectively.

The idea behind the standard notion of QEC is to encode the logical system into a suitable code subspace in such a way that the logical information is effectively 'hidden' by entanglement and thus remains recoverable under certain noise. Owing much to the clean yet powerful scheme of stabilizer codes[2], it is customary in the study of QEC to seek and understand quantum codes that enable exact recovery. However, generalized notions of codes that may only achieve QEC in an approximate manner could be adequate for practical purposes and outperform exact QEC codes in various ways[6,7]. Furthermore, they encompass a much broader range of scenarios especially innate in physical contexts, underscoring the fundamental importance of approximate quantum error correction (AQEC) from both practical and theoretical perspectives.

Despite the extensive study of QEC, our knowledge on AQEC codes is limited to various scattered situations (for example, examples in spin chains[8], covariant codes[9–16] and quasi-exact codes[17,18]), with their fundamental understanding remaining elusive. It is worth mentioning a striking finding that an extremely small imprecision tolerance suffices to enable the decoding radius to match the code distance[7,19], in stark contrast with the situation of exact QEC, signifying the intrinsic

[1]Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada. [2]Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada. [3]Joint Center for Quantum Information and Computer Science (QuICS) and Computer Science Department, University of Maryland, College Park, MD, USA. [4]Yau Mathematical Sciences Center, Tsinghua University, Beijing, China. e-mail: zwliu0@tsinghua.edu.cn

https://doi.org/10.1038/s41567-024-02621-x

distinction in nature. In the literature, the notion of AQEC commonly just means that the imprecision is vanishingly small in system size. However, this can even be naturally achieved by a trivial encoding defined by appending a series of garbage states to the logical state for random local noise, simply because the chance of logical information being affected is vanishingly small (Methods and Supplementary Note IV). This suggests that our current understanding of AQEC is too coarse.

In addressing this predicament, we establish a general theory of AQEC codes based on quantum circuit complexity whose importance permeates quantum computation, complexity theory and physics[20–23], encompassing scenarios both with and without geometric locality. More specifically, we define a code parameter called subsystem variance that characterizes the fluctuation of marginals of the physical system and is closely connected with the existing notions of AQEC imprecision and many-body entanglement. We derive critical values of the subsystem variance that scale roughly as $O(k/n)$, below which the entire code subspace is subject to non-trivial circuit complexity lower bounds depending on the geometry. The conditions are nearly optimal in certain regimes and provide meaningful criteria for interesting codes in general, as supported by concrete examples. From a code perspective, our results suggest that it is reasonable to consider $O(k/n)$ a boundary between subspaces that should be regarded as 'acceptable' AQEC codes and those that should not be. Our theory offers not only a fundamental understanding of non-trivial AQEC codes but also useful methods for the widely important but notoriously difficult problem of proving circuit lower bounds. The wide applicability of our theory is demonstrated by various examples arising from both quantum computation and physics.

Remarkably, our AQEC framework and results have broad applications in physics, bridging general information-theoretic properties with quantum physical features in many ways. In particular, we gain new insights into non-trivial quantum order or long-range entanglement that underpin 'exotic' quantum features including topological order and criticality that are of great importance in condensed-matter and high-energy physics' contexts. For topological order, we demonstrate that AQEC offers a unifying framework for rigorously understanding the relationship between strict notions of gapped topological order and the long-range entanglement and topological entanglement entropy (TEE) signatures. For critical quantum systems, we show that a power-law AQEC imprecision is a fundamental nature of the CFT codes that emerge at low energies and discuss how our theory may provide insights into quantum gravity through AdS/CFT. It is notable that a roughly $O(1/n)$ imprecision scaling naturally arises as some kind of 'threshold' in several different situations.

Circuit complexity from AQEC

Here we will work with multiqubit systems, namely, those living in a Hilbert space given by the tensor product of multiqubit Hilbert spaces. The notions of locality and geometry associated with such a many-body system are captured by an adjacency graph, the edges of which define the connection relations among the nodes (qubits). Two prototypical types of adjacency graph are complete (all-to-all) graphs and local lattices embedded in finite spatial dimensions, with the former being more common in computer science contexts and the latter incorporating geometric locality that is usually essential in physical contexts (Fig. 1 shows an illustration).

Now, we lay the groundwork for our study of AQEC. We propose to call any 2^k-dimensional subspace of an n-qubit Hilbert space an $((n, k))$ quantum code ($k < n$) as it represents an encoding of a k-qubit logical system into an n-qubit physical system, and any pure state within this subspace is called a code state. Of course, it may not be a good QEC code as QEC requires intricate structures. A theme of this work is to understand the meaning of the deviation from ideal QEC codes, which is generic and serves as the basis for the theory for AQEC. This deviation can naturally be quantified by how well the recovery can restore the

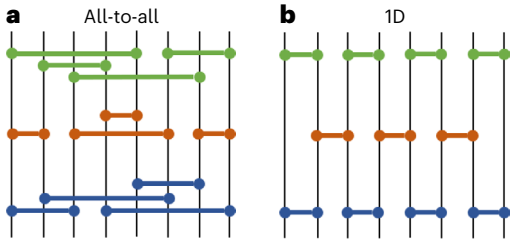

Fig. 1 | Locality and quantum circuit complexity. In a circuit, the local gates are only allowed to involve connected nodes with respect to the adjacency graph. Characterizing the time required for implementation, the depth of a circuit is given by the number of gate layers, where each layer can only consist of disjoint gates. **a**,**b**, Illustrations of all-to-all (**a**) and one-dimensional (1D) (**b**) 2-local circuits, associated with the complete graph and 1D chain; both examples are depth-3 circuits. The all-to-all (quantum circuit) complexity of an n-qubit quantum state $|\Psi\rangle$, denoted as $\mathscr{C}(\psi)$, is defined as the minimum depth of the 2-local quantum circuits that generate $|\Psi\rangle$ from $|0\rangle^{\otimes n}$. The geometric (quantum circuit) complexity of $|\Psi\rangle$, denoted as $\mathscr{C}_{\mathrm{G}_D}(\psi)$, is defined analogously, with the difference being that only two-qubit gates acting on the nearest neighbours in a D-dimensional integer lattice G_D are allowed. We also consider the δ-robust versions of these complexities, denoted by $\mathscr{C}^\delta$, which are defined as the minimum corresponding complexity of any state $|\Psi'\rangle$ within the δ vicinity of $|\Psi\rangle$, that is, $\||\Psi'\rangle - |\Psi\rangle\|_1 \leq \delta$.

logical information after the system undergoes noise; specifically, for encoding channel $\mathcal{E}$ and some noise channel $\mathcal{N}$, the QEC inaccuracy is defined as

$$\tilde{\varepsilon}(\mathcal{N}, \mathcal{E}) := \min_{\mathcal{R}} P\left(\mathcal{R} \circ \mathcal{N} \circ \mathcal{E}, \mathrm{id}_{\mathrm{L}}\right), \tag{1}$$

that is, the minimum distance (here we adopt the channel purified distance P; Methods provides the detailed definition) between the overall logical channel after recovery $\mathcal{R}$ and the logical identity id_{L}. From what features of the code does such QEC inaccuracy originate? To understand this, we introduce a type of parameter intrinsically associated with the code space $\mathfrak{C}$ (image of $\mathcal{E}$) that we call subsystem variance:

$$\varepsilon_{\mathrm{G}}(\mathfrak{C}, d) := \max_{\psi \in \mathfrak{C}, |S| \leq d} \|\psi_S - \Gamma_S\|_1, \tag{2}$$

where $\Gamma := \frac{1}{2^k} \sum_{i=1}^{2^k} |\psi_i\rangle\langle\psi_i|$ is the statistical average of $\{\psi_i\}$ that spans $\mathfrak{C}$, that is, the maximally mixed state of $\mathfrak{C}$, and S is a connected (local) subsystem with respect to the adjacency graph G (subscript S denotes the reduced state on S). Here d should be treated as a tunable variable that generalizes the notion of code distance. Intuitively, the subsystem variance limits the accessible information from the subsystems and is thus closely tied to entanglement and QEC properties. In particular, it bounds the violation of the Knill–Laflamme QEC conditions[24] and broadly characterizes the QEC inaccuracy. As an extreme instance, under the same locality restriction,

$$\frac{\varepsilon}{4} \leq \tilde{\varepsilon} \leq 2^{k/2}\sqrt{\varepsilon} \tag{3}$$

for any noise represented by replacement channels, including, for example, erasure, complete depolarizing and reset channels (some relations between QEC error and the violation of Knill–Laflamme conditions are known[25,26]). A complete form of this result can be found in Methods, with full proofs given in Supplementary Note I. In addition, we can also establish two-way bounds that relate subsystem variance with coherent information, a well-known quantity that characterizes quantum information loss[27–29] (Supplementary Note II). The physics discussions in later sections mainly concern scenarios where ε and $\tilde{\varepsilon}$ convey similar messages. They may generally be referred to as code/AQEC error at appropriate instances.

We now introduce our main results on quantum circuit complexity from AQEC for both all-to-all and geometric cases (Fig. 1 shows an illustration and brief definitions of circuit complexity, with further details provided in Methods). The log symbols denote the logarithm to base 2. $H_2(p) = -p\log p - (1-p)\log(1-p)$ is the binary entropy function; whenever it appears, it is assumed that $p < 1/2$.

Theorem 1. *Given an* $((n,k))$ *code* $\mathfrak{C}$*, the* δ*-robust all-to-all quantum circuit complexity of any code state* $|\psi\rangle \in \mathfrak{C}$ *satisfies* $\mathscr{C}^{\delta}(\psi) > \log d$*, if* $H_2(\varepsilon_{\mathrm{g}}(\mathfrak{C},d)/2 + \delta/2) < k/n$ *with* g *being the complete graph* (ε *is defined with respect to any* d *qubits*)*, where* $\varepsilon + \delta < 1$.

Theorem 2. *Given an* $((n,k))$ *code* $\mathfrak{C}$*, the* δ*-robust geometric circuit complexity with respect to adjacency graph* G_D *embedded in a* D*-dimensional integer lattice of any code state* $|\psi\rangle \in \mathfrak{C}$ *satisfies* $\mathscr{C}^{\delta}_{\mathrm{G}_D}(\psi) > (d^{1/D}-1)/2$*, if* $H_2(\varepsilon_{\mathrm{G}_D}(\mathfrak{C},d)/2 + \delta/2) < k/n$*, where* $\varepsilon + \delta < 1$.

Evidently, our results cover exact QEC codes as special cases (for example, recovering the stabilizer code result of ref. 30 in the no low-energy trivial state context (NLTS) and the long-range entangled property of the toric code), which can be seen by noting that $\varepsilon_{\mathrm{G}}(\mathfrak{C}, d-1) = 0$ for any $[[n,k,d]]$ code and any adjacency graph G so that our code error conditions are automatically satisfied. It is also worth emphasizing that the above results are independent of the topology of the base manifold, and reflect universal complexity features of the entire code spaces that encompass arbitrary superpositions of special wavefunctions in the code, which are important but rarely understood in physics contexts. From the perspective of proving circuit complexity lower bounds, our approach can be used to establish bounds for specific states beyond the applicability of the theorems. A physically interesting family of examples is given by what we call momentum codes, which are discussed in detail in Supplementary Note VI. Another noteworthy point is that AQEC properties are able to guarantee the intrinsic all-to-all circuit complexity, which is not constrained by geometry and spatial locality. Finally, note that our results conversely indicate lower bounds on code error depending on the code states (Supplementary Note III).

Although our results apply to any specific n, one is usually most interested in the asymptotic scalings in the thermodynamic (large-n) limit. The distinction of whether the circuit complexity of a system is $O(1)$ (that is, finite in the large-n limit) holds exceptional importance in both physics and complexity theory. States with $\omega(1)$ (superconstant) complexity, often referred to as long-range entangled states when a proper notion of geometric locality is present, are generally associated with certain kinds of non-trivial quantum order and play central roles in the theory of phases of matter and Hamiltonian complexity (in both contexts, it is sometimes desirable to consider $\tilde{\omega}(1)$ complexity); however, this difference is inconsequential in our results). The key implications of our theory are summarized in the following corollary.

Corollary 3. *Given an* $((n,k))$ *code* $\mathfrak{C}$ *with subsystem variance* $\varepsilon_{\mathrm{G}}(\mathfrak{C},d)$ *where* $d = \omega(1)$*. Suppose* $H_2(\varepsilon_{\mathrm{G}}(\mathfrak{C},d)/2) < k/n$*, which is satisfied particularly when*

- $k = \tilde{O}(1), \varepsilon = \tilde{o}(1/n)$;
- $k = \Omega(n), \varepsilon = o(1)$.

Then, for any code state $|\psi\rangle \in \mathfrak{C}$*, it holds that* $\mathscr{C}_{\mathrm{G}}(\psi) = \omega(1)$.

Figure 2 depicts the schematic of the circuit complexity 'phase diagrams' for any d and G in terms of the corresponding ε as well as $\tilde{\varepsilon}$ (for replacement channels) over k.

Most importantly, our results identify 'non-trivial' regimes of code parameters—any state that belongs to a code within these non-trivial regimes is subject to our circuit complexity lower bounds.

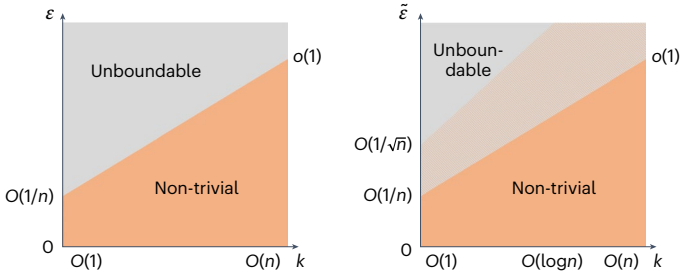

Fig. 2 | Schematic circuit complexity phase diagrams of general quantum codes. 'Non-trivial' and 'unboundable' mean that our complexity bounds hold for any code state, or are inapplicable, respectively. The diagram for $\tilde{\varepsilon}$ is obtained using equation (3); the intermediate regime can be regarded as an expanded boundary where the applicability of the complexity bounds depends on the code.

Note that the critical scalings below which our bounds can apply can be roughly achieved in a naive manner, suggesting that our code error conditions for circuit lower bounds represent meaningful conditions for AQEC codes. Besides, the conditions are nearly tight for small k, as evident from the Heisenberg chain code. This is further discussed in Methods, with full details given in the Supplementary Information.

The universality of our framework enables applications in an exceptionally broad range of scenarios in coding theory and physics. In Table 1, we summarize the properties of various representative types of AQEC code originating from diverse contexts, which also provide meaningful examples for different parameter and complexity regimes. The CFT codes and momentum codes are introduced and analysed in this work.

Topological order

Our complexity results shed a new light on topological order, a widely studied concept in modern condensed-matter physics that characterizes exotic quantum phases of matter arising from many-body entanglement. A central problem in the study of topological order is to identify simple criteria or indicators for states associated with systems having topological order. As signified by the prototypical example of the toric code[4], QEC is a representative feature (and application) of topologically ordered systems. Indeed, QEC properties underlie the well-established topological quantum order (TQO) condition, which is tied to strong physical notions of topological order such as gap stability[31,32], essentially demanding that the state belong to an (almost) exact QEC code with macroscopic (at least poly(n)) distance. On the other hand, based directly on many-body entanglement properties, there are two other prominent characteristics, oftentimes considered definitions, for states with topological order: long-range entanglement[21,33] and TEE[34,35]. Despite extensive study and usage of all the three conditions, their relationship has not been systematically understood. We now demonstrate that AQEC provides a general framework that allows us to rigorously compare the TQO and entanglement conditions, thereby sharpening our understanding of topological order. Here we present an overview, with more details given in Methods and Supplementary Note VII.

First, a direct implication of our results is a general quantitative understanding of the gap between TQO and long-range entanglement conditions, with the former being strictly stronger than the latter. To be more specific, recall that long-range entanglement means superconstant geometric circuit complexity. According to Corollary 3, the code property requirements in TQO can be relaxed from exponentially small error under macroscopic distance[32] to $\varepsilon = \tilde{o}(1/n)$ under any superconstant (for example, logarithmic) distance, still ensuring that all the code states are long-range entangled. This is a substantial relaxation that is expected to encompass wide-ranging physical situations. However, in the literature, the notions of long-range entanglement and topological order are frequently lumped together, especially for

Table 1 | Various representative examples of AQEC codes

Code	k	ε^{a}	Code-space complexity
ETH energy window	$\Omega(n)$	$\exp(-O(n))$	Non-trivial*
Random unitary code[b]	$\Omega(n)$	$\exp(-O(n))$	Non-trivial*
Topological code	$O(1)$	$\exp(-O(n))$	Non-trivial
'Good' AQLDPC code	$\tilde{\Omega}(n)$	$O(1/\mathrm{polylog}(n))$	Non-trivial*
Heisenberg chain code	$\Omega(\log n)$	$\tilde{\omega}(1/n)$	Unboundable
CFT low-energy sector	$O(1)$	$\Theta(1/n^{\Delta/D})$[c]	Indefinite*
Momentum code	$O(\log n)$	$O(1)$	Unboundable

[a] With respect to suitable superconstant d. [b] With $1-\exp(-O(n))$ probability. [c] Δ is the minimum scaling dimension of the CFT. The ETH and Heisenberg chain codes (Supplementary Note V provides a refined analysis) are defined in ref. 8; the good approximate quantum low-density parity-check (LDPC) code specifically refers to the spacetime Hamiltonian construction in ref. 52; the italicized codes are explicitly studied in this paper. The rightmost column lists the complexity results deduced from our theory, where the asterisks specify applicability to all-to-all complexity and the remaining entries concern geometric complexity in their respective native dimensions.

gapped systems. Our observation elucidates the discrepancy between the two notions in terms of AQEC parameters.

Next, let us consider TEE, which widely serves as a simple information-theoretic signature for topological order. More concretely, consider the standard two-dimensional (2D) setting, where the ground states of gapped systems commonly obey an area law, with possible subleading corrections originating from long-range entanglement inherent in topologically ordered systems, based on which TEE is defined. For general AQEC code states, we show the following:

Proposition 4. *(Informal) Consider an* $((n,k))$ *code with an area-law code state defined on a 2D torus. Suppose* $\varepsilon = o(1/n)$ *for any contractible region of linear size; then, all the code states have non-trivial TEE.*

Specifically, the Abelian topological order saturates the best TEE lower bound of $k/2$ *from our approach.*

As a corollary, a 2D area-law state with trivial TEE does not belong to any code that achieves $\varepsilon = o(1/n)$ *on linear-size contractible regions.*

To conclude, a key takeaway is that TEE and long-range entanglement have a similar ~$1/n$ robustness against AQEC error. It is worth emphasizing that our discussion applies to any subspace, not hinging on gapped ground spaces (associated with Hamiltonians) as conventional in the context of topological order.

Critical systems and CFTs

Critical quantum systems, widely described by CFT, represent another prominent type of quantum order with wide-ranging physical importance. The nature of critical systems, specifically their gaplessness and scale invariance, suggests a universal presence of highly non-trivial entanglement that supports interesting quantum coding properties. In particular, in the context of quantum gravity, the concept of CFT codes is expected to play a pivotal role, in light of the fundamental connection[5,36] between QEC and AdS/CFT correspondence[37,38].

Physically, it is most natural to consider the low-energy sectors as code spaces, where k does not scale with the system size and all the states are CFT states. As explained in greater detail in Methods and Supplementary Note VIII, they generally give rise to intrinsic AQEC codes whose properties are closely connected to the physics of the system and can be concretely analysed by employing techniques from the field of CFT. Indeed, for $d = O(\log\log(n))$ on a complete graph, ε follows a power-law scaling with the exponent determined by the minimum scaling dimension Δ, precisely

$$\epsilon = \tilde{\Theta}(n^{-\Delta/D}). \tag{4}$$

Crucially, this shows that the polynomial AQEC error is a fundamental nature of CFT, different from, for example, topological order.

As per Corollary 3, our theory has the following implication that could be of particular interest in relation to holography: if the minimum scaling dimension $\Delta > D$ (such that $\varepsilon = \tilde{o}(1/n)$), then any state in the CFT code is not just long-range entangled (as can also be inferred from the general spatial correlation properties of CFT using Lieb–Robinson-type arguments[31,39]), but actually has the fundamentally stronger feature of being intrinsically non-trivial, that is, has superconstant all-to-all complexity. This all-to-all property is morally congruent with the ultrastrong long-range interactions and chaotic behaviours (think, for example, the Sachdev–Ye–Kitaev model), which are strong physical signatures of non-trivial gravity duals in AdS/CFT[40–44], providing new insights into the duality from a quantum information perspective. Indeed, a large scaling dimension is associated with a large central charge[45,46] and bulk field mass[38,47], which are in accordance with strong coupling and complexity. Although our understanding of their relationship is incomplete, the connections among all these perspectives indicates that AQEC plays a profound role in the physics of CFT and gravity that could be fruitful to further study.

Remarkably, the threshold of our complexity results' applicability, $\Delta = D$, is of special physical importance as global symmetry current operators have scaling dimension D. Therefore, if the system has a continuous global symmetry, Δ is capped at D so that the CFT code obeys $\varepsilon = \Omega(1/n)$, thereby falling outside the regime of universal non-trivial complexity. As an implication, AQEC parameters could be used to probe symmetries. This is consistent with the scaling limit of the AQEC error of covariant codes[10–15,48] and highlights the fundamental nature of the ~$1/n$ boundary from yet another angle. In addition, $\Delta > D$ implies a fundamental tension with global symmetries on both sides of AdS/CFT[49], which could be a situation of special importance given the key role of symmetries in quantum gravity[49–51]. All things considered, our discussions suggest compelling motivations to look for and study CFTs in the $\Delta > D$ regime. This problem is non-trivial and interesting by itself because such a theory is not forbidden by known constraints in any D (note that the stress–energy tensor has scaling dimension $D+1$), but we have limited knowledge of natural examples.

Discussion and outlook

We studied general quantum code subspaces that do not necessarily enable exact QEC, offering a systematic understanding of their non-triviality. We proved code error thresholds below which circuit complexity lower bounds for any code state remain robust, for any geometry. Through the examples of topological order and critical systems that respectively represent gapped and gapless order, we have demonstrated that AQEC provides a powerful unifying lens for understanding the physics of complex many-body quantum systems, highlighting the value of insights from quantum information in physics.

A noteworthy phenomenon is that intrinsically approximate codes with power-law error scaling, although highly atypical in the sense that randomly constructed codes almost always exhibit exponentially small errors, naturally arise in wide-ranging physical scenarios like gapless systems in fundamental ways. In particular, error scaling near the ~$1/n$ boundary often emerges hand in hand with symmetries or specific structures of states. Based on our theory, the non-trivial order associated with such scenarios, which we call marginal order, is expected to represent a general type of order that is fundamentally distinct from topological order in its stability and other physical properties. It would be interesting to further investigate this notion. Note that the approximate Eastin–Knill theorems[10–15,48] place codes with

continuous transversal gates or symmetries outside the acceptable code regime, further strengthening our understanding of the incompatibility between symmetry and QEC.

The CFT codes represent a family of intrinsically AQEC codes that warrant deeper investigation, especially because of their importance to the understanding of quantum criticality as well as quantum gravity. Specifically, our preliminary discussion points towards several interesting avenues for more rigorous consideration, such as the existence of non-trivial gravity duals and implications for symmetries in quantum gravity. Furthermore, it is potentially valuable to consider extensions to continuous variable and fermionic systems, open quantum systems and more general QEC settings.

To conclude, a key takeaway is that we expect AQEC codes to substantially extend the scope and utility of conventional notions of QEC in the realms of practical quantum technologies, complexity theory and physics. We hope this study sparks further exploration in AQEC and its physical as well as practical applications.

Online content

Any methods, additional references, Nature Portfolio reporting summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41567-024-02621-x.

References

1. Shor, P. W. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A* **52**, R2493–R2496 (1995).
2. Gottesman, D. *Stabilizer Codes and Quantum Error Correction*. PhD thesis, California Institute of Technology (1997).
3. Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (Cambridge Univ. Press, 2000).
4. Kitaev, A. Y. Fault-tolerant quantum computation by anyons. *Ann. Phys.* **303**, 2–30 (2003).
5. Almheiri, A., Dong, X. & Harlow, D. Bulk locality and quantum error correction in AdS/CFT. *J. High Energy Phys.* **2015**, 163 (2015).
6. Leung, D. W., Nielsen, M. A., Chuang, I. L. & Yamamoto, Y. Approximate quantum error correction can lead to better codes. *Phys. Rev. A* **56**, 2567–2573 (1997).
7. Crépeau, C., Gottesman, D. & Smith, A. D. Approximate quantum error-correcting codes and secret sharing schemes. In *Eurocrypt* 285–301 (Springer, 2005).
8. Brandão, F. G. S. L., Crosson, E., Şahinoğlu, M. B. & Bowen, J. Quantum error correcting codes in eigenstates of translation-invariant spin chains. *Phys. Rev. Lett.* **123**, 110502 (2019).
9. Hayden, P., Nezami, S., Popescu, S. & Salton, G. Error correction of quantum reference frame information. *PRX Quantum* **2**, 010326 (2021).
10. Faist, P. et al. Continuous symmetries and approximate quantum error correction. *Phys. Rev. X* **10**, 041018 (2020).
11. Woods, M. P. & Alhambra, Á. M. Continuous groups of transversal gates for quantum error correcting codes from finite clock reference frames. *Quantum* **4**, 245 (2020).
12. Kubica, A. & Demkowicz-Dobrzański, R. Using quantum metrological bounds in quantum error correction: a simple proof of the approximate Eastin–Knill theorem. *Phys. Rev. Lett.* **126**, 150503 (2021).
13. Yang, Y., Mo, Y., Renes, J. M., Chiribella, G. & Woods, M. P. Optimal universal quantum error correction via bounded reference frames. *Phys. Rev. Res.* **4**, 023107 (2022).
14. Zhou, S., Liu, Z.-W. & Jiang, L. New perspectives on covariant quantum error correction. *Quantum* **5**, 521 (2021).
15. Liu, Z.-W. & Zhou, S. Approximate symmetries and quantum error correction. *NPJ Quantum Inf.* **9**, 119 (2023).
16. Kong, L. & Liu, Z.-W. Near-optimal covariant quantum error-correcting codes from random unitaries with symmetries. *PRX Quantum* **3**, 020314 (2022).
17. Wang, D.-S., Zhu, G., Okay, C. & Laflamme, R. Quasi-exact quantum computation. *Phys. Rev. Res.* **2**, 033116 (2020).
18. Wang, D.-S., Wang, Y.-J., Cao, N., Zeng, B. & Laflamme, R. Theory of quasi-exact fault-tolerant quantum computing and valence-bond-solid codes. *New J. Phys.* **24**, 023019 (2022).
19. Bergamaschi, T., Golowich, L. & Gunn, S. Approaching the quantum singleton bound with approximate error correction. Preprint at https://doi.org/10.48550/arXiv.2212.09935 (2022).
20. Aaronson, S. The complexity of quantum states and transformations: from quantum money to black holes. Preprint at https://doi.org/10.48550/arXiv.1607.05256 (2016).
21. Wen, X.-G. Topological order: from long-range entangled quantum matter to an unification of light and electrons. *ISRN Condens. Matter Phys.* **2013**, 198710 (2013).
22. Susskind, L. Computational complexity and black hole horizons. *Fortschr. Phys.* **64**, 24–43 (2016).
23. Brown, A. R. & Susskind, L. Second law of quantum complexity. *Phys. Rev. D* **97**, 086015 (2018).
24. Knill, E. & Laflamme, R. Theory of quantum error-correcting codes. *Phys. Rev. A* **55**, 900–911 (1997).
25. Bény, C. & Oreshkov, O. General conditions for approximate quantum error correction and near-optimal recovery channels. *Phys. Rev. Lett.* **104**, 120501 (2010).
26. Ng, H. K. & Mandayam, P. Simple approach to approximate quantum error correction based on the transpose channel. *Phys. Rev. A* **81**, 062342 (2010).
27. Schumacher, B. & Nielsen, M. A. Quantum data processing and error correction. *Phys. Rev. A* **54**, 2629–2635 (1996).
28. Lloyd, S. Capacity of the noisy quantum channel. *Phys. Rev. A* **55**, 1613–1622 (1997).
29. Schumacher, B. & Westmoreland, M. D. Approximate quantum error correction. Preprint at https://doi.org/10.48550/arXiv.quant-ph/0112106 (2001).
30. Anshu, A. & Nirkhe, C. Circuit lower bounds for low-energy states of quantum code Hamiltonians. In *Leibniz International Proceedings in Informatics (LIPIcs)*. Vol. 215, 6:1–6:22 (Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2022).
31. Bravyi, S., Hastings, M. B. & Verstraete, F. Lieb-Robinson bounds and the generation of correlations and topological quantum order. *Phys. Rev. Lett.* **97**, 050401 (2006).
32. Bravyi, S., Hastings, M. B. & Michalakis, S. Topological quantum order: stability under local perturbations. *J. Math. Phys.* **51**, 093512 (2010).
33. Chen, X., Gu, Z.-C. & Wen, X.-G. Local unitary transformation, long-range quantum entanglement, wave function renormalization and topological order. *Phys. Rev. B* **82**, 155138 (2010).
34. Kitaev, A. & Preskill, J. Topological entanglement entropy. *Phys. Rev. Lett.* **96**, 110404 (2006).
35. Levin, M. & Wen, X.-G. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.* **96**, 110405 (2006).
36. Harlow, D. TASI lectures on the emergence of the bulk in AdS/CFT. Preprint at https://doi.org/10.48550/arXiv.1802.01040 (2018).
37. Maldacena, J. M. The large-*N* limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.* **38**, 1113–1133 (1999).
38. Witten, E. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.* **2**, 253–291 (1998).
39. Lieb, E. H. & Robinson, D. W. The finite group velocity of quantum spin systems. *Commun. Math. Phys.* **28**, 251–257 (1972).
40. Kitaev, A. A simple model of quantum holography. https://online.kitp.ucsb.edu/online/entangled15/kitaev/ (2015).
41. Maldacena, J. & Stanford, D. Remarks on the Sachdev-Ye-Kitaev model. *Phys. Rev. D* **94**, 106002 (2016).

42. Heemskerk, I., Penedones, J., Polchinski, J. & Sully, J. Holography from conformal field theory. Preprint at https://doi.org/10.1088/1126-6708/2009/10/079 (2009).
43. Maldacena, J., Shenker, S. H. & Stanford, D. A bound on chaos. *J. High Energy Phys.* **2016**, 106 (2016).
44. Witten, E. Three-dimensional gravity revisited. Preprint at https://doi.org/10.48550/arXiv.0706.3359 (2007).
45. Hellerman, S. A universal inequality for CFT and quantum gravity. *J. High Energy Phys.* **2011**, 130 (2011).
46. Rattazzi, R., Rychkov, S. & Vichi, A. Central charge bounds in 4D conformal field theory. *Phys. Rev. D* **83**, 046011 (2011).
47. Gubser, S. S., Klebanov, I. R. & Polyakov, A. M. Gauge theory correlators from non-critical string theory. *Phys. Lett. B* **428**, 105–114 (1998).
48. Liu, Z.-W. & Zhou, S. Quantum error correction meets continuous symmetries: fundamental trade-offs and case studies. Preprint at https://doi.org/10.48550/arXiv.2111.06360 (2023).
49. Harlow, D. & Ooguri, H. Symmetries in quantum field theory and quantum gravity. *Commun. Math. Phys.* **383**, 1669–1804 (2021).
50. Misner, C. W. & Wheeler, J. A. Classical physics as geometry. *Ann. Phys.* **2**, 525–603 (1957).
51. Banks, T. & Seiberg, N. Symmetries and strings in field theory and gravity. *Phys. Rev. D* **83**, 084019 (2011).
52. Bohdanowicz, T. C., Crosson, E., Nirkhe, C. & Yuen, H. Good approximate quantum LDPC codes from spacetime circuit Hamiltonians. In *Proc. 51st Annual ACM SIGACT Symposium on Theory of Computing* 481–490 (ACM, 2019).

Methods

Asymptotic notations

In addition to the standard Bachmann–Landau notation using O, o, Ω, ω and Θ symbols[53], we shall also use the 'soft' notation with a tilde on top, which hide polylogarithmic factors that are unimportant in our context; explicitly, for $A \in \{O, \Omega, \Theta\}$, $\tilde{A}(f(n))$ means $A(f(n)\mathrm{polylog}(n))$ for some polylog function, and for $a \in \{o, \omega\}$, $\tilde{a}(f(n))$ means $a(\mathrm{f}(n)\mathrm{polylog}(n))$ for any polylog function.

Some formal definitions

Here we provide the full mathematical definitions of several key concepts used in this work for readers' convenience.

Trace norm and channel purified distance. Let $\|O\|_1$ denote the trace norm of operator O given by $\|O\|_1 := \mathrm{Tr}\sqrt{O^\dagger O}$. The purified distance between quantum states ρ and σ is defined as

$$P(\rho,\sigma) := \sqrt{1-f(\rho,\sigma)^2}, \tag{5}$$

where f is the Uhlmann fidelity, which is given as

$$f(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}. \tag{6}$$

Then, the proper channel version of the purified distance, known as the completely bounded purified distance, between two quantum channels $\mathcal{M}_1$ and $\mathcal{M}_2$, is defined as

$$P(\mathcal{M}_1,\mathcal{M}_2) := \sqrt{1-F(\mathcal{M}_1,\mathcal{M}_2)^2} \tag{7}$$

$$= \max_\rho P\left((\mathcal{M}_1\otimes \mathrm{id})(\rho),(\mathcal{M}_2\otimes \mathrm{id})(\rho)\right), \tag{8}$$

where F is the completely bounded fidelity of channels given by

$$F(\mathcal{M}_1,\mathcal{M}_2) := \min_\rho f((\mathcal{M}_1\otimes \mathrm{id})(\rho),(\mathcal{M}_2\otimes \mathrm{id})(\rho)), \tag{9}$$

with the optimization running over input states on any extended system.

Note that the channel purified distance is chosen due to its desirable properties but other channel distance measures like Bures and diamond distances can also be considered and will yield similar results.

Quantum circuit complexity. Generally, for an n-qubit quantum state $|\Psi\rangle$, the (quantum circuit) complexity associated with the adjacency graph G, denoted by $\mathscr{C}_\mathrm{G}(\psi)$, is defined as the minimum depth (number of layers) of 2-local (with respect to G) quantum circuits that generate $|\Psi\rangle$ from $|0\rangle^{\otimes n}$. More precisely,

$$\mathscr{C}_\mathrm{G}(\psi) := \min\left\{l : |\psi\rangle = \prod_{i=1}^{l} U_i|0\rangle^{\otimes n}\right\}, \tag{10}$$

where the U_i values must be a tensor product of disjoint two-qubit unitary gates acting on the nearest neighbours in G. The two standard scenarios specifically discussed in the main text are all-to-all quantum circuit complexity corresponding to complete graph g and geometric quantum circuit complexity corresponding to finite-dimensional lattices G_D. The δ-robust versions of these complexities, denoted by $\mathscr{C}^\delta$, are defined as the minimum corresponding complexity of any state $|\Psi'\rangle$ within the δ vicinity of $|\Psi\rangle$ in trace norm, namely,

$$\mathscr{C}^\delta_\mathrm{G}(\psi) := \min_{\||\psi'\rangle-|\psi\rangle\|_1\le\delta} \mathscr{C}_\mathrm{G}(\psi'). \tag{11}$$

Relating subsystem variance and QEC inaccuracy

Here we present a detailed form of the two-way relation between the subsystem variance and QEC inaccuracy given in equation (3).

Proposition 5. *Let $\check{\mathcal{N}}_R$ be any replacement channel acting on a d-qubit subsystem R that is connected with respect to adjacency graph* G. *Denote the overall channel by $\mathcal{N} = \check{\mathcal{N}}_R \otimes \mathrm{id}_{\bar{R}}$. Consider the subsystem variance for a particular subsystem R defined as $\varepsilon_\mathrm{G}(\mathfrak{C},R) := \max_{\sigma\in\mathfrak{C}} \|\sigma_R - \Gamma_R\|_1$. It holds that*

$$\frac{1}{4}\varepsilon_\mathrm{G}(\mathfrak{C},R) \le \bar{\varepsilon}(\mathcal{N},\mathcal{E}) \le 2^{k/2}\sqrt{\varepsilon_\mathrm{G}(\mathfrak{C},R)}. \tag{12}$$

A version for the overall subsystem variance is directly obtained by optimizing over R:

$$\frac{1}{4}\varepsilon_\mathrm{G}(\mathfrak{C},d) \le \max_R \bar{\varepsilon}(\mathcal{N},\mathcal{E}) \le 2^{k/2}\sqrt{\varepsilon_\mathrm{G}(\mathfrak{C},d)}. \tag{13}$$

This result is established based on methods provided in another work[25] that relate QEC inaccuracy and violation of Knill–Laflamme conditions, further using complementary channel methods and information-theoretic bounds. Full details of the proof are given in Supplementary Note I. Note that the bounds here assume replacement channels, which can be regarded as the worst class of noise channels. Modified bounds for general noise channels can also be derived.

Remarks on generalizations of the setup

Several assumptions in the setup and definitions are only made for the convenience of exposition and can be generalized in various ways that may encompass a wide range of physically relevant scenarios.

- The qubit assumption is not essential; generalizations to higher local dimensions are straightforward.
- The methods to derive circuit complexity bounds can be generalized to arbitrary graphs (shown below and Supplementary Note III).
- Generalizing 2-locality to t-locality for any finite t only introduces constant factors to the results; more explicitly, the proofs reveal that if we consider t-local gates instead of 2-local gates, then our circuit complexity bounds for the all-to-all and geometric cases hold with an extra $1/\log t$ factor and an extra $1/(t-1)$ factor, respectively.
- Several physical variants of the setting, including quasi-local gates with fast decaying tails, quasi-adiabatic evolutions[54] and more general lattices in certain dimensions, are expected to retain the relevant messages in this work.

Further remarks on the complexity results

Our results are applicable to different notions of AQEC error. Recall that the code error conditions for circuit complexity lower bounds in the main text take the form $H_2(\varepsilon_\mathrm{G}(\mathfrak{C},d)/2 + \delta/2) < k/n$. Using equation (3) (Proposition 5), the conditions can be alternatively expressed in terms of QEC inaccuracy $\bar{\varepsilon}$, substituting $\varepsilon_\mathrm{G}(\mathfrak{C},d)$ with $4\bar{\varepsilon}(\mathcal{N},\mathcal{E})$ for replacement channels $\mathcal{N}$ acting on any d-local subsystems with respect to the associated adjacency graph G.

Additionally, it is worth noting that our complexity results indicate 'intrinsic' circuit complexities of code states themselves, which should not be confused with the depth of the encoding circuits that are blind to the input logical states.

We now briefly discuss two natural examples that shed light on the non-triviality of our complexity characterizations of codes.

First, the critical scalings of code error below which our complexity bounds apply can be roughly achieved in a naive manner, by considering a trivial kind of 'encoding' map that simply appends garbage states to the original logical state, that is, $\mathcal{E}(|\psi\rangle) = |\psi\rangle\otimes|\mathrm{gar}\rangle\otimes\ldots\otimes|\mathrm{gar}\rangle$, where $\langle\psi|\mathrm{gar}\rangle = 0$, which we call redundant encoding. This suggests that our code error conditions for circuit lower bounds represent meaningful conditions for AQEC codes. A detailed analysis can be found in Supplementary Note IV.

Moreover, the code error thresholds for our complexity bounds are nearly optimal in the sense that they cannot be much improved at least in the regime of small k. This is seen from the Heisenberg chain code[8], where the code states are well-separated ground states of the ferromagnetic Heisenberg chain. Such codes have AQEC errors close to the critical scaling for small k, yet their code space contains states with low circuit complexity, signifying the near-tightness of the error thresholds. Details can be found in Supplementary Note V.

Complexity theorem for general adjacency graphs

As mentioned, our approach can be generalized to obtain circuit complexity bounds for arbitrary adjacency graphs. The general theorem takes the following form:

Theorem 6. *Given an* $((n,k))$ *code* $\mathfrak{C}$*, the* δ*-robust circuit complexity with respect to adjacency graph* G *of any code state* $|\psi\rangle \in \mathfrak{C}$ *satisfies* $\mathscr{C}^{\delta}_{\mathrm{G}}(\psi) > f^{-1}(d)$*, where* $f(t)$ *is the maximum size of the light cone of a single qubit under depth-*t *circuits, if* $H_2(\varepsilon_{\mathrm{G}}(\mathfrak{C},d)/2+\delta/2) < k/n$*, where* $\varepsilon+\delta<1$.

For certain adjacency graphs including D-dimensional lattices, the results can be further improved by exploiting the structure of the graph (Supplementary Note III).

Intuition for the complexity theorems for AQEC codes

With improved forms of the results and detailed proofs provided in Supplementary Note III, we now distil the core intuitions for our code-space complexity theorems, which generally apply to any connectivity, from all-to-all to geometric cases.

Our results roughly say the 'distance' (noise size) under which a code can maintain a sufficiently small code error indicates circuit complexity lower bounds. An overall conceptual message is that higher complexity is generally associated with smaller code errors and larger code rates. The main proof idea, adapting a method in ref. 30 to our AQEC setting, goes as follows. Suppose a code state is generated by a circuit Q of some low depth from $|0\rangle^{\otimes n}$ where each qubit's effects are confined within its light cone determined by Q. Now run this circuit backwards (apply $Q^{\dagger}$) on the maximally mixed code state Γ. Using the properties of the light cone, one finds that if ε within the light cone scale of Q ($\exp(O(n))$ for all-to-all circuits and $O(n^D)$ for D-dimensional circuits) is small, then the output of the backward circuit is well approximated by $|0\rangle$ locally and therefore the entire system has small entropy due to subadditivity. Making ε sufficiently small leads to contradictions with the entropy of Γ directly determined by k, which, in turn, implies complexity lower bounds because the light cones are too small for consistency.

More on topological order

We first introduce, in more detail, the three major characteristics of topological order discussed in the main text, namely, TQO conditions, long-range entanglement and TEE.

The TQO condition[31,32] asserts that the ground subspace is an (almost exact) QEC code with a macroscopic distance that grows as a positive power of the lattice size L. Usually, the ground subspace is assumed to be an exact code, up to at most exponentially small corrections.

Long-range entanglement[21,33] means that the ground state of the system cannot be smoothly transformed into a product state by any finite-depth local unitary circuit.

TEE[34,35] is an entropic measure of the topological contribution to entanglement. Specifically, suppose the entanglement entropy of any contractible subsystem A takes the form

$$S(A) = al(A) - \gamma + o(1), \tag{14}$$

where the first term manifests the area law, with a being some constant and $l(A)$ being the length of the boundary of A, and the correction γ is the TEE, which is expected to be a universal constant signifying topological order.

We would like to demonstrate that TQO is notably stronger than the other two characteristics: roughly speaking, although the TQO condition requires the AQEC error to be $\exp(-O(n))$, an $O(1/n)$ error is sufficient for the presence of long-range entanglement and non-trivial TEE.

The comparison between TQO and long-range entanglement is discussed in the main text. Here we present the formal result for TEE and provide some intuitions with full details given in Supplementary Note VII.

Proposition 7. *Consider an* $((n,k))$ *code defined on a 2D lattice on a torus. Suppose that* $\varepsilon = o(1/n)$ *for any contractible region of size* d*, and there exists a code state with area-law entanglement. Then, in the thermodynamic limit, the TEE of any code state satisfies*

$$\gamma \geq k/\max\{2, 2\lfloor n/2d \rfloor\}. \tag{15}$$

Specifically, we have the best bound $\gamma \geq k/2$*, which is saturated by an Abelian topological order, if the code conditions hold for*

i. $d > n/4$*, or*
ii. *any* d *linear in* n *if, additionally, for error regions that do not contain non-contractible loops on the torus,* $\tilde{\varepsilon} = o(1/n)$ *can be achieved by recovery operations acting within the* $O(1)$ *distance to the error region* ($\ell = O(1)$*, where the* ℓ *parameter is defined elsewhere*[55]).

This result is proven using the prescription from ref. 56. The main idea is to apply the Markov entropy decomposition[57] to relate k and a signed sum of subregion entropies in which all the area-law terms are cancelled out, leaving only TEE with corrections due to subsystem variance. When $\varepsilon = o(1/n)$, the corrections turn out to be vanishingly small, ensuring non-trivial bounds on TEE. Conversely, when $\varepsilon = \omega(1/n)$, there is no non-trivial bound because TEE cannot overshadow the corrections. Note that in Proposition 7, d can be improved to cn with any $c > 0$ if we further require a local recoverability feature (which is expected to generally hold for topological order). This is proven by leveraging the expansion lemma from ref. 55. We further note that this result does not hinge on a strict area law, that is, small fluctuations in correction γ are allowed, in which case the lower bound is for the average TEE. Supplementary Note VII provides detailed proofs.

By considering a deformation of the toric code through adding string tension (such that the code states are string-net wavefunctions with tension)[58], we can construct a physically interesting example of a code family with tunable AQEC error where the TEE vanishes as the AQEC error increases, in accordance with our results (Supplementary Note VII).

AQEC from CFTs

Here we briefly introduce the ideas behind the analysis of AQEC properties of CFT codes using techniques native to the field of CFT (Supplementary Note VIII provides the full details).

Consider the low-energy sectors as code spaces, where k does not scale with the system size and all the states are CFT states. Let the system be defined on a hypersphere S^D of $D \in \mathbb{Z}^+$ spatial dimensions. Using the state-operator correspondence[59] on the cylinder geometry $S^D \times R$, it can be shown that the one-point functions $\langle\phi_\beta|\phi_\alpha|\phi_\gamma\rangle$ for code states $|\phi_{\beta,\gamma}\rangle$ and local ($O(1)$-size) primary operator ϕ_α, which can be related to the Knill–Laflamme conditions for QEC, exhibit the scaling behaviour[60,61]

$$\langle\phi_\beta|\,\phi_\alpha\,|\phi_\gamma\rangle = \Theta\left(\frac{1}{n^{\Delta_\alpha/D}}\right), \tag{16}$$

where n is the total system size and Δ_α is the scaling dimension of ϕ_α. These one-point functions are studied for extracting the conformal data in the CFT literature[62–65]. Here we can use them to compute the

AQEC error scaling. For the sake of non-trivial circuit complexity arguments, consider sufficiently small subsystems (that are not necessarily spatially local) of size $O(\log\log(n))$. The reduced code states on the subsystem can then be expanded with an orthonormal operator basis, which can be well approximated by a product of local operators via proper renormalization group flow so that the scaling in equation (16) is expected to hold up to minor (at most polylog(n)) factors. Then, it can be shown (Supplementary Note VIII) that ε follows a power-law scaling with the exponent determined by the minimum scaling dimension Δ, precisely,

$$\varepsilon = \tilde{\Theta}(n^{-\Delta/D}). \tag{17}$$

Data availability

No data have been generated in this work.

References

53. Knuth, D. E. Big omicron and big omega and big theta. *SIGACT News* **8**, 18–24 (1976).
54. Hastings, M. B. & Wen, X.-G. Quasiadiabatic continuation of quantum states: the stability of topological ground-state degeneracy and emergent gauge invariance. *Phys. Rev. B* **72**, 045141 (2005).
55. Flammia, S. T., Haah, J., Kastoryano, M. J. & Kim, I. H. Limits on the storage of quantum information in a volume of space. *Quantum* **1**, 4 (2017).
56. Kim, I. H. Long-range entanglement is necessary for a topological storage of quantum information. *Phys. Rev. Lett.* **111**, 080503 (2013).
57. Poulin, D. & Hastings, M. B. Markov entropy decomposition: a variational dual for quantum belief propagation. *Phys. Rev. Lett.* **106**, 080403 (2011).
58. Chen, X., Zeng, B., Gu, Z.-C., Chuang, I. L. & Wen, X.-G. Tensor product representation of a topological ordered phase: necessary symmetry conditions. *Phys. Rev. B* **82**, 165119 (2010).
59. Simmons-Duffin, D. TASI lectures on the conformal bootstrap. Preprint at https://doi.org/10.48550/arXiv.1602.07982 (2016).
60. Cardy, J. L. Conformal invariance and universality in finite-size scaling. *J. Phys. A* **17**, L385 (1984).
61. Cardy, J. L. Letter to the Editor—universal amplitudes in finite-size scaling: generalisation to arbitrary dimensionality. *Curr. Phys. Sources Comments* **2**, 370–373 (1988).
62. Cardy, J. L. Operator content of two-dimensional conformally invariant theories. *Nucl. Phys. B* **270**, 186–204 (1986).
63. Blöte, H. W. J., Cardy, J. L. & Nightingale, M. P. Conformal invariance, the central charge and universal finite-size amplitudes at criticality. *Phys. Rev. Lett.* **56**, 742–745 (1986).
64. Zou, Y., Milsted, A. & Vidal, G. Conformal fields and operator product expansion in critical quantum spin chains. *Phys. Rev. Lett.* **124**, 040604 (2020).
65. Hu, L., He, Y.-C. & Zhu, W. Operator product expansion coefficients of the 3D Ising criticality via quantum fuzzy spheres. *Phys. Rev. Lett.* **131**, 031601 (2023).

Acknowledgements

We thank A. Anshu, Y.-C. He, H. Ma, S. Sang, B. Yoshida, S. Zhou, Z. Zhou and Y. Zou for valuable discussions and feedback. J.Y. would like to thank A. Burkov for his support. D.G. is partially supported by the National Science Foundation (RQS QLCI grant OMA-2120757). Z.-W.L. is partially supported by a startup funding at YMSC, Tsinghua University. Research at the Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada, and by the Province of Ontario through the Ministry of Colleges and Universities.

Author contributions

Z.-W.L. conceived and designed this project, drawing inspiration from conversations with J.Y. and W.Y. and especially D.G. J.Y. led the technical development of this work, which was supervised by Z.-W.L. All authors contributed to the discussions that shaped this work. J.Y. and Z.-W.L. are primarily responsible for the technical content and writing of the paper.

Competing interests

The authors declare no competing interests.

Additional information

Supplementary information The online version contains supplementary material available at https://doi.org/10.1038/s41567-024-02621-x.

Correspondence and requests for materials should be addressed to Zi-Wen Liu.

Peer review information *Nature Physics* thanks Michael Kastoryano and the other, anonymous, reviewer(s) for their contribution to the peer review of this work.