Tutorial: Device-independent random number generation

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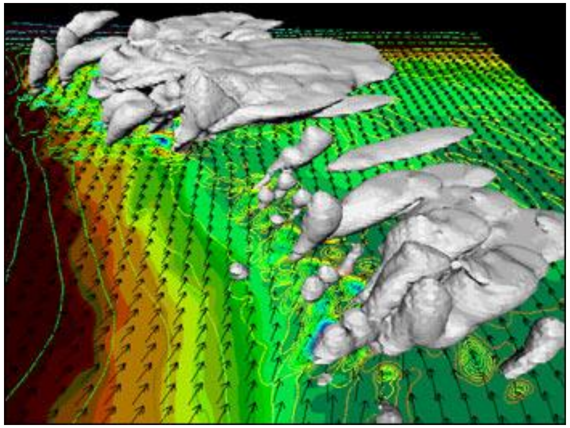
Outline

- Brief motivation of random number generation
- Discuss what we mean by a random number
- Discuss some ways of generating them leading up to device-independent protocols
- Explain the main ideas behind a device-independent random number generator
- Discuss what it means for a protocol to be secure
- Briefly mention related tasks

Why are random numbers important?

gambling | simulations | cryptography



Random number generation

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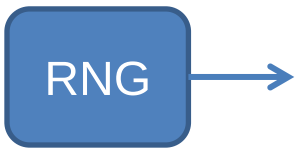
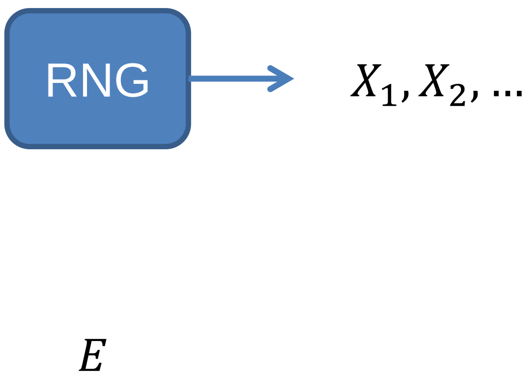
What is a random number?



$X_1, X_2, \ldots$

- Unpredictable by anyone (independent of everything else)
- Uniformly distributed

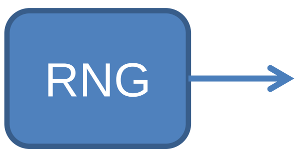
What is a random number?



- More formally, we can say

  X_j is a uniform random bit (with respect to E) if $P_{X_j|E} = P_{X_j} = \frac{1}{2}$ where E represents 'everything else' (includes $X_1, \ldots, X_{j-1}$)

What is a random number?



$X_1, X_2, \dots$

E

- Quantum case

$$\sum_x \frac{1}{|X|} |x\rangle\langle x|_A \otimes \rho_E$$

What do we want in a random number generation protocol?

- Secure
- Reliable
- Easy to implement
  - Technologically feasible
  - Requires few devices
- Have a fast rate

Security

- Protocol should come with a rigorous, precisely formulated security proof and statement of validity
  - E.g., if the protocol is used correctly, then no adversary can learn the random numbers even given unlimited time/resources (unless physics is wrong)

Security

- Protocol should come with a rigorous, precisely formulated security proof and statement of validity
  - E.g., if the adversary is limited to have particular computational resources, the random string can be treated as random for a certain amount of time.

How might we generate random numbers?

How might we generate random numbers? Classical case

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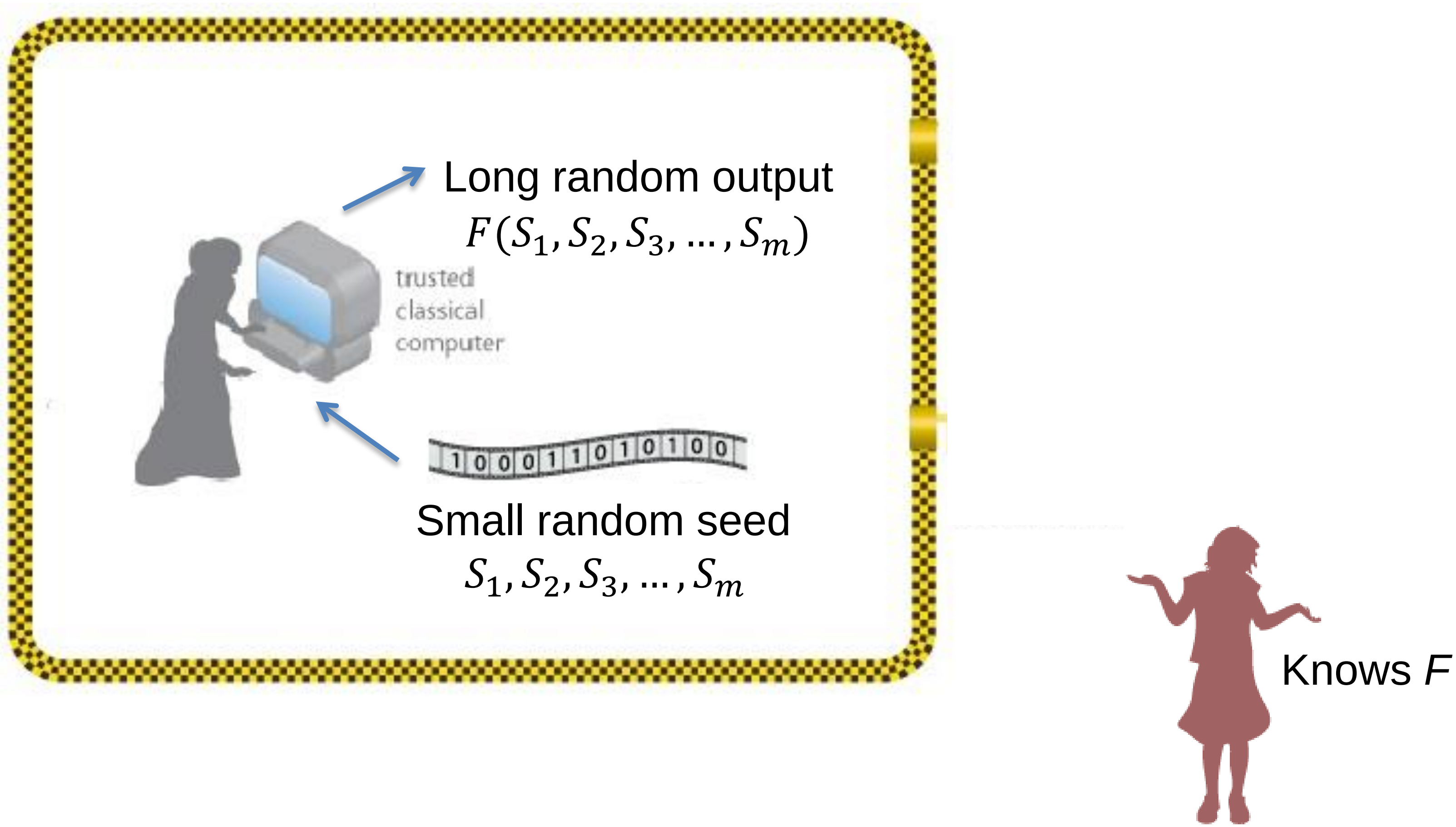
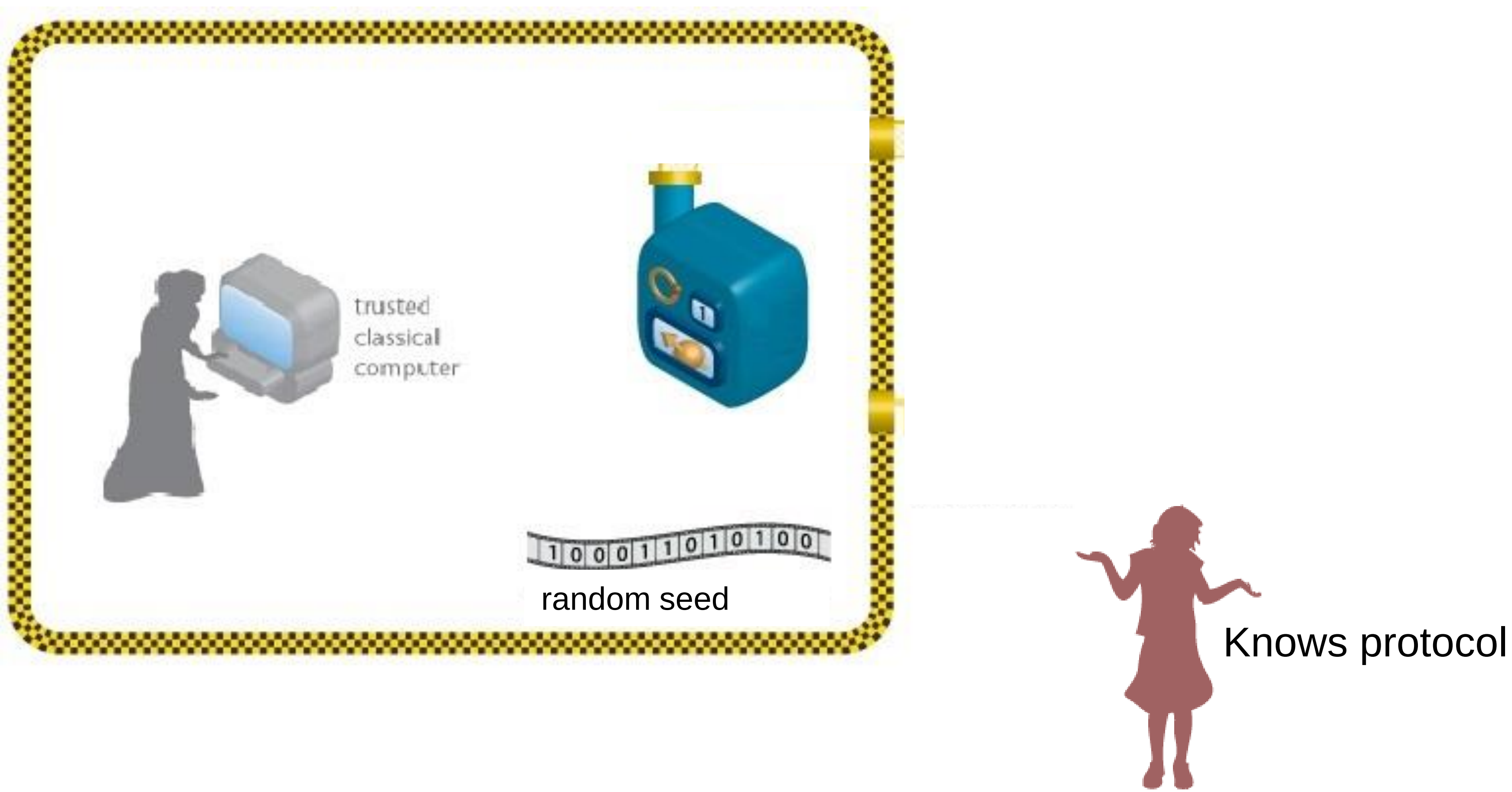


Classical case

Drawbacks:

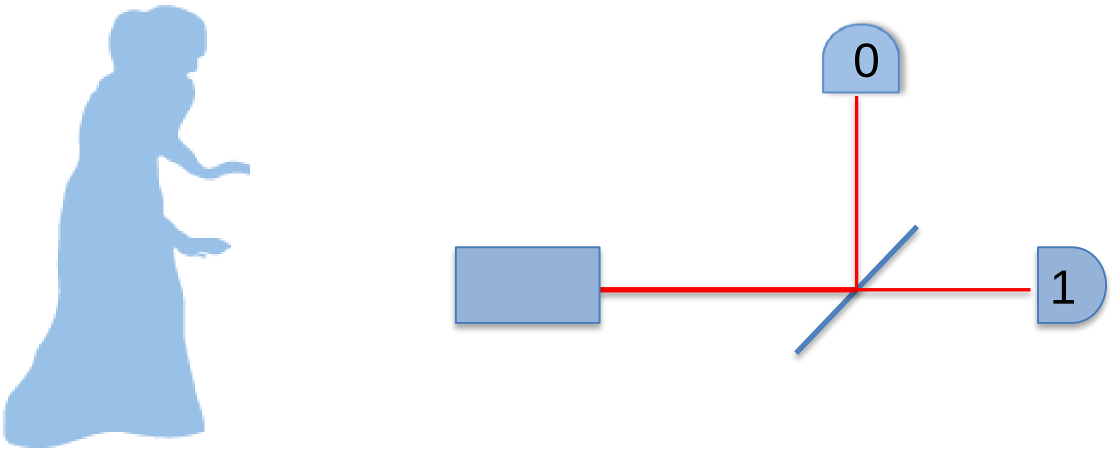
- Cannot have unconditional security
- In general, we cannot prove hardness of breaking the protocol

Trusted quantum case



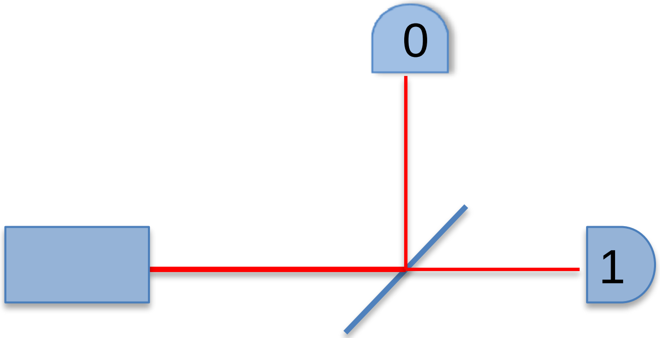
Trusted quantum case

For example: use a beamsplitter



Trusted quantum case

For example: use a beamsplitter

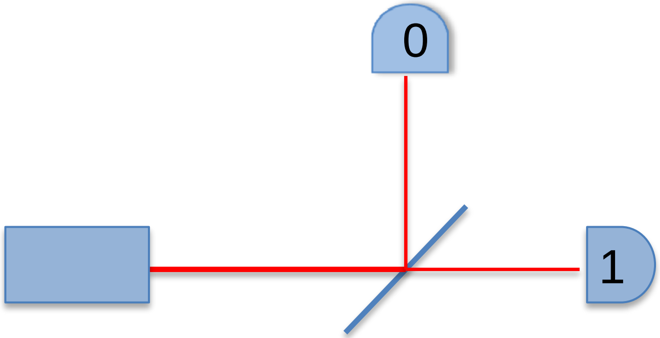


This might be ok if:

- Trust the equipment
- Ensure that it doesn't change over time

Trusted quantum case

For example: use a beamsplitter

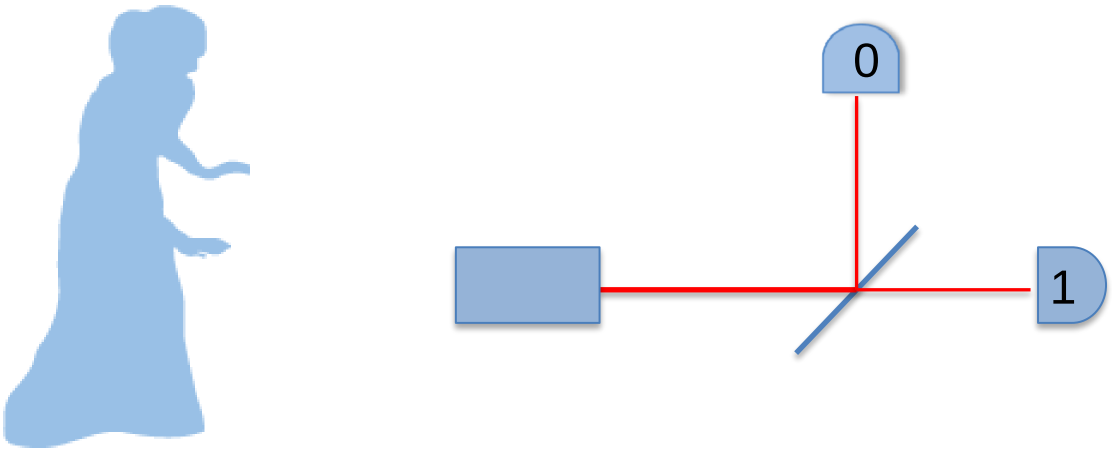


This might be ok if:

- Trust the equipment
- Ensure that it doesn't change over time
- (Trust the physics and that it is complete)

Trusted quantum case

For example: use a beamsplitter



Ideally we would like a certificate that outputs are random

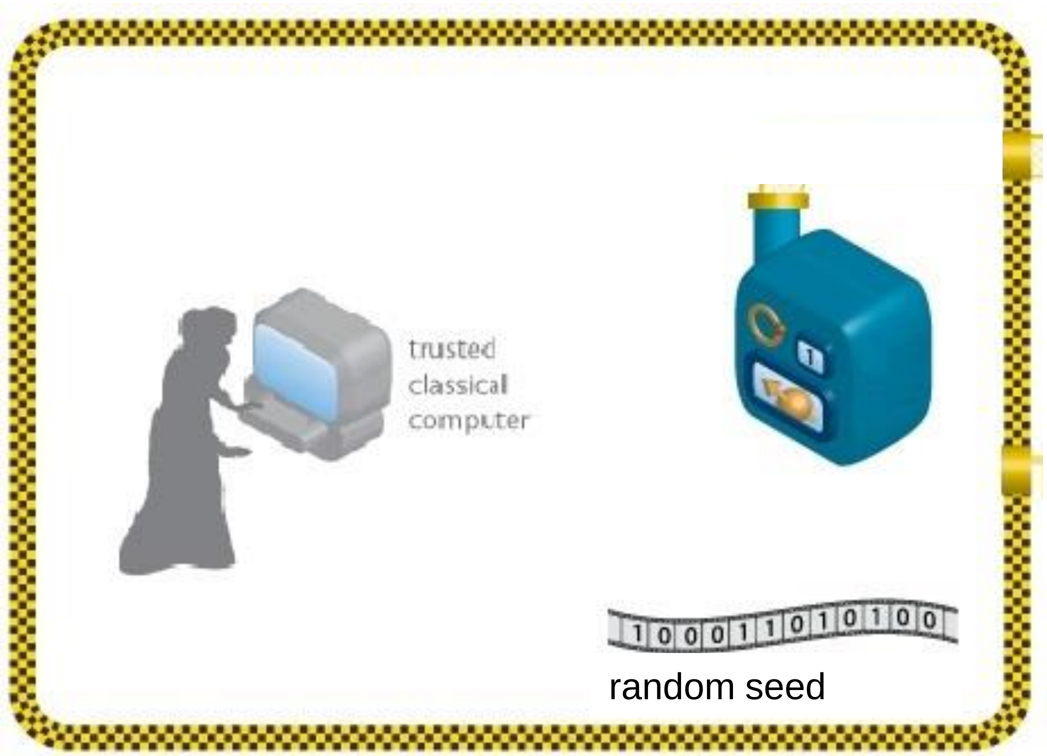
Trusted quantum case

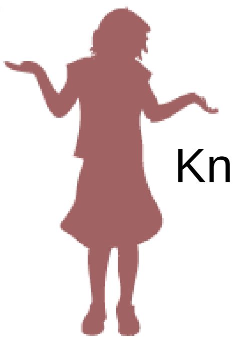
Removes classical drawbacks; in particular, can have security based on physics.

New drawbacks:

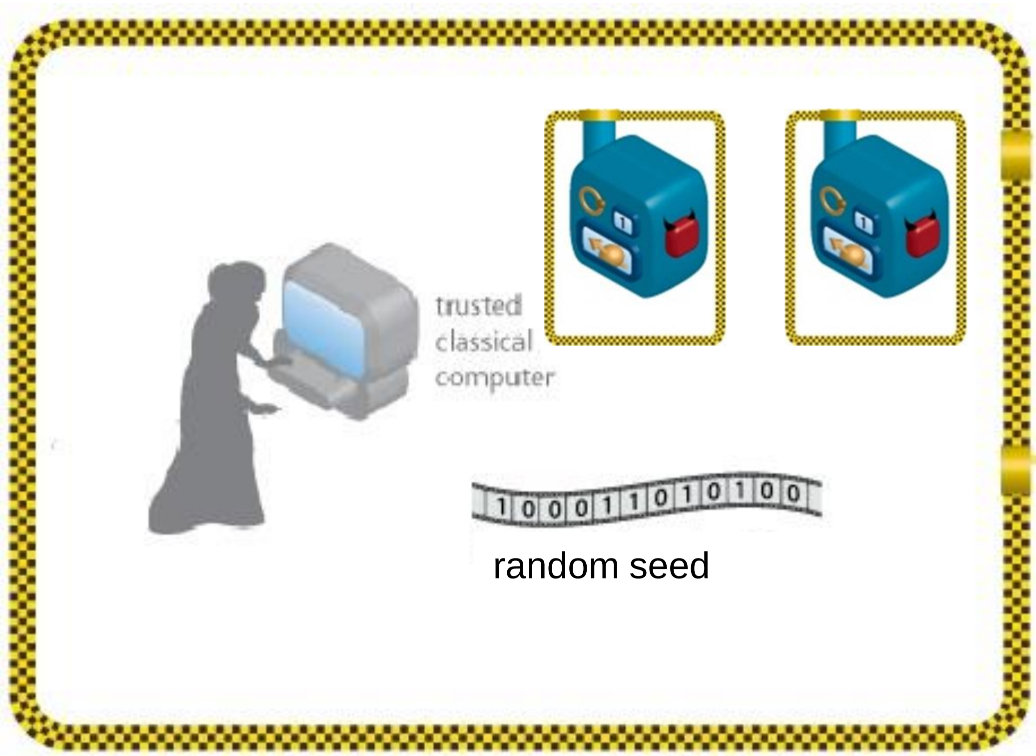
- Technologically harder to implement (but not too bad)
- Security relies on the devices behaving correctly

The setup (quantum)





The setup (device-independent)

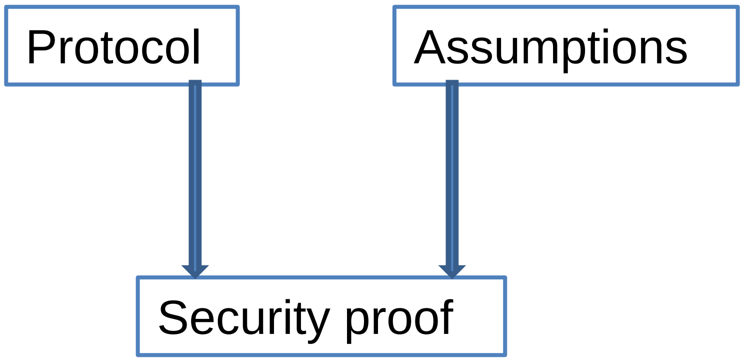


Want to generate longer random string

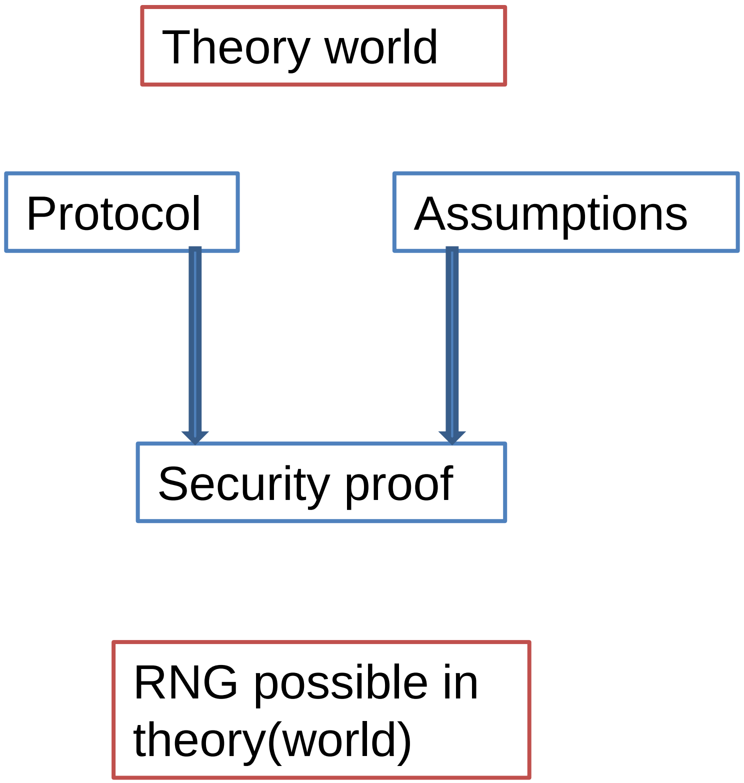
Device-independence

- No assumptions made about the workings of the devices used
- However, we do need some assumptions, in particular, both strong lab walls and initial randomness [necessary for cryptography]

Security proofs



Security proofs



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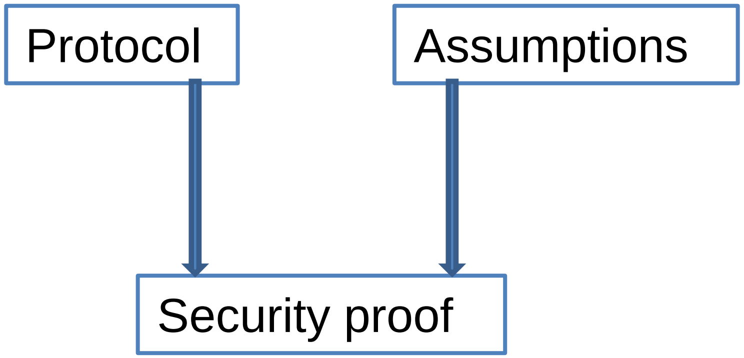
Security proofs

Theory world

Real world



Is our theory world proof relevant in the real world?

RNG possible in theory(world)

Security proofs

Weaker assumptions → More security

Security proofs

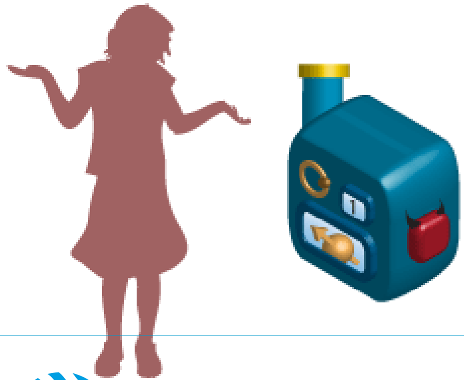
Weaker assumptions → More security

- Device-independence tries to remove all the assumptions on the devices
- Removes this mismatch problem between the real world and theory world

Security proofs

Weaker assumptions → More security

- No assumptions on devices means the security proof has to work even with maliciously constructed devices.



Security proofs

Weaker assumptions → More security

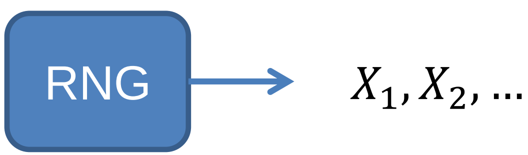
- Protocol remains secure if devices stop working properly or are tampered with

- Protocol checks the workings of the devices on-the-fly (hence, self-testing)

Device-independence: main ideas

- Don’t trust devices, so have to test them

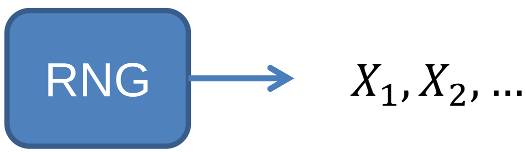
How can we test the devices?



How can we test for randomness?

- **Overlapping permutations:** Analyse sequences of five consecutive random numbers. The 120 possible orderings should occur with statistically equal probability.
- **Ranks of matrices:** Select some number of bits from some number of random numbers to form a matrix over {0,1}, then determine the rank of the matrix. Count the ranks.
- **Monkey tests:** Treat sequences of some number of bits as "words". Count the overlapping words in a stream. The number of "words" that don't appear should follow a known distribution.
- **The craps test:** Play 200,000 games of craps, counting the wins and the number of throws per game. Each count should follow a certain distribution.

How can we test for randomness?



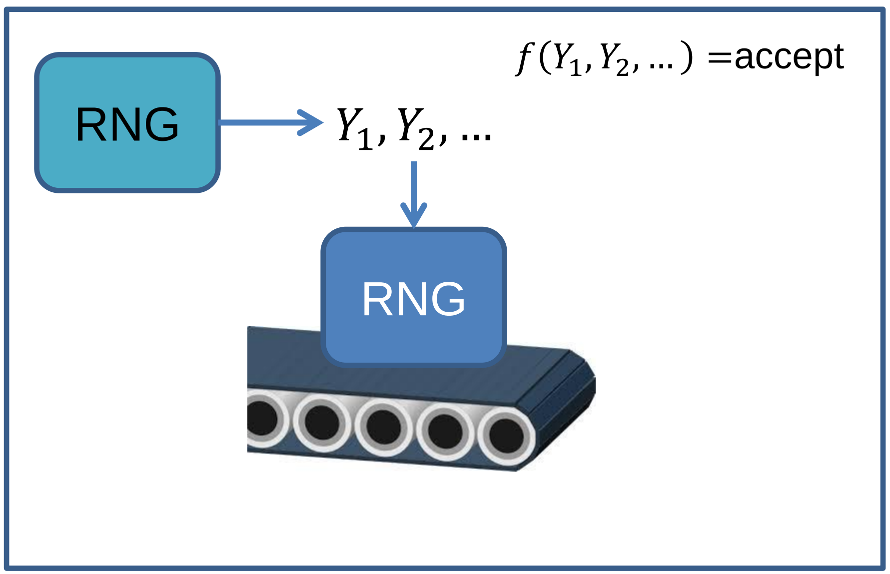
- There is no good test that acts only on the outputs.

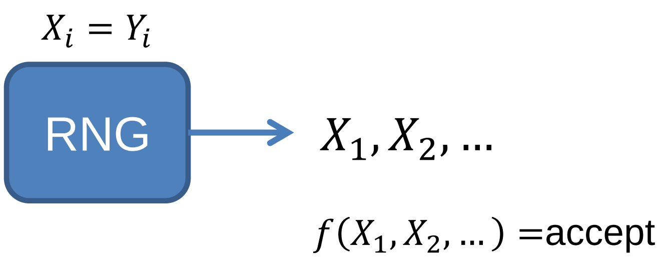
- No f such that

$$f(X_1, X_2, \ldots) = \begin{cases} \text{accept} \\ \text{reject} \end{cases}$$

with accept only if the sequence is random.

How can we test for randomness?





More advanced test

- There is no good test with only one device

$X_1, X_2, \ldots$

$A_1, A_2, \ldots$ (Random)

$$f(A_1, A_2, \ldots, X_1, X_2, \ldots) \in \{\text{pass, fail}\}$$

Adversary knows f
Adversary can supply pre-programmed classical device that will always pass

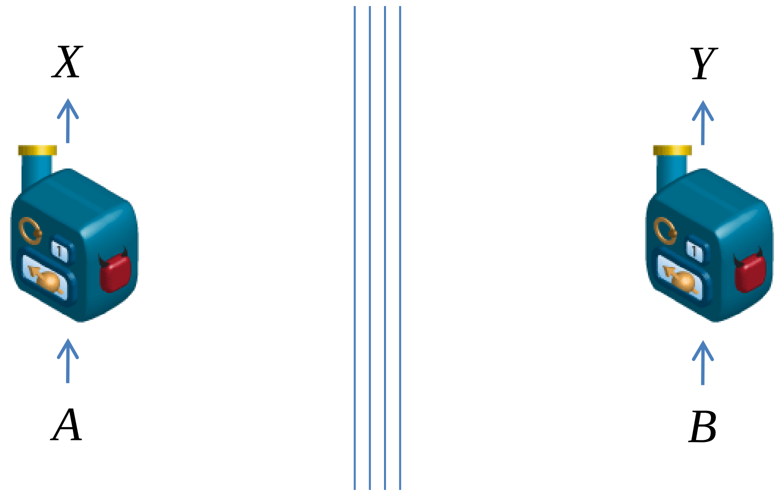
Device-independent randomness expansion: main ideas

Bell inequality violation → Non-classical behaviour

(loophole-free)

Device-independent randomness expansion: main ideas

- Bell-inequality violation



$P_{XY|AB}$ violates a Bell inequality
A and B random
Devices cannot communicate

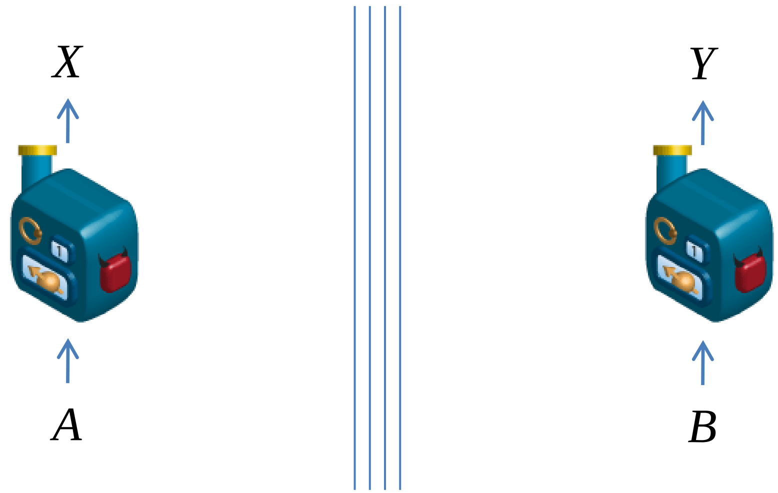
Bell's theorem

Eve cannot know X
X not function of A

Roughly the idea of Ekert 91, although note that we're not making key here

Device-independent randomness expansion: main ideas

- Bell-inequality violation



$P_{XY|AB}$ violates a Bell inequality
A and B random
Devices cannot communicate

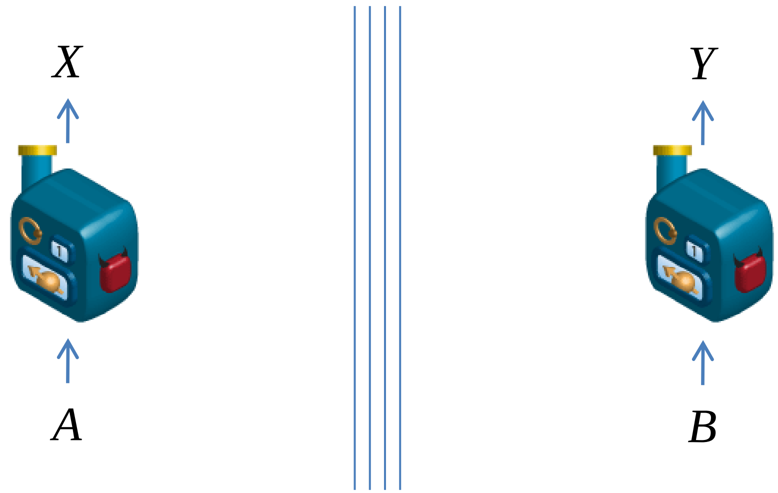
Bell's theorem

Eve cannot know X
X not function of A

- Doesn't mean that X is perfectly random

Device-independent randomness expansion: main ideas

- Bell-inequality violation



$P_{XY|AB}$ violates a Bell inequality
A and B random
Devices cannot communicate

↓ Bell's theorem

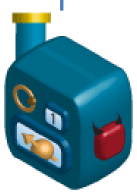
Eve cannot know X
X not function of A

- E.g. CHSH game winning probability

Device-independent randomness expansion: main ideas

- CHSH game

$X \in \{0,1\}$



$A \in \{0,2\}$

$Y \in \{0,1\}$

$B \in \{1,3\}$

Win if
$X = Y$ for $(\mathrm{A}, \mathrm{B}) = (0,1), (2,1)$ or $(2,3)$
$X \neq Y$ for $(A, B) = (0,3)$.

- $P_{cl} \leq \frac{3}{4}$ $\qquad$ $P_{qm} \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.85.$

(Bell value 2) $\qquad$ (Bell value $2\sqrt{2}$)

Device-independent randomness expansion: main ideas

$X \in \{0,1\}$

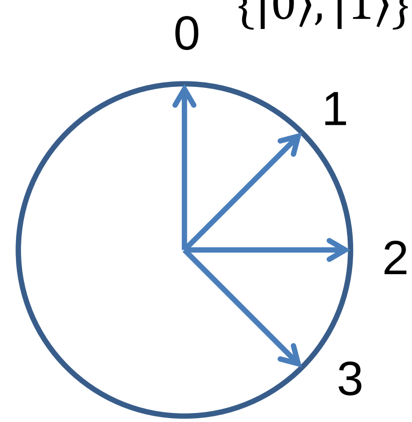
$A \in \{0,2\}$

$Y \in \{0,1\}$

$B \in \{1,3\}$

Win if
$X = Y$ for $(\mathrm{A}, \mathrm{B}) = (0,1), (2,1)$ or $(2,3)$
$X \neq Y$ for $(A, B) = (0,3)$.

$\{|0\rangle, |1\rangle\}$



$\{|+\rangle, |-\rangle\}$

- $P_{qm} \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.85$

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Device-independent randomness expansion: main ideas

Maximum quantum violation → Devices share max entangled (pure) state

↓

No entanglement with Eve

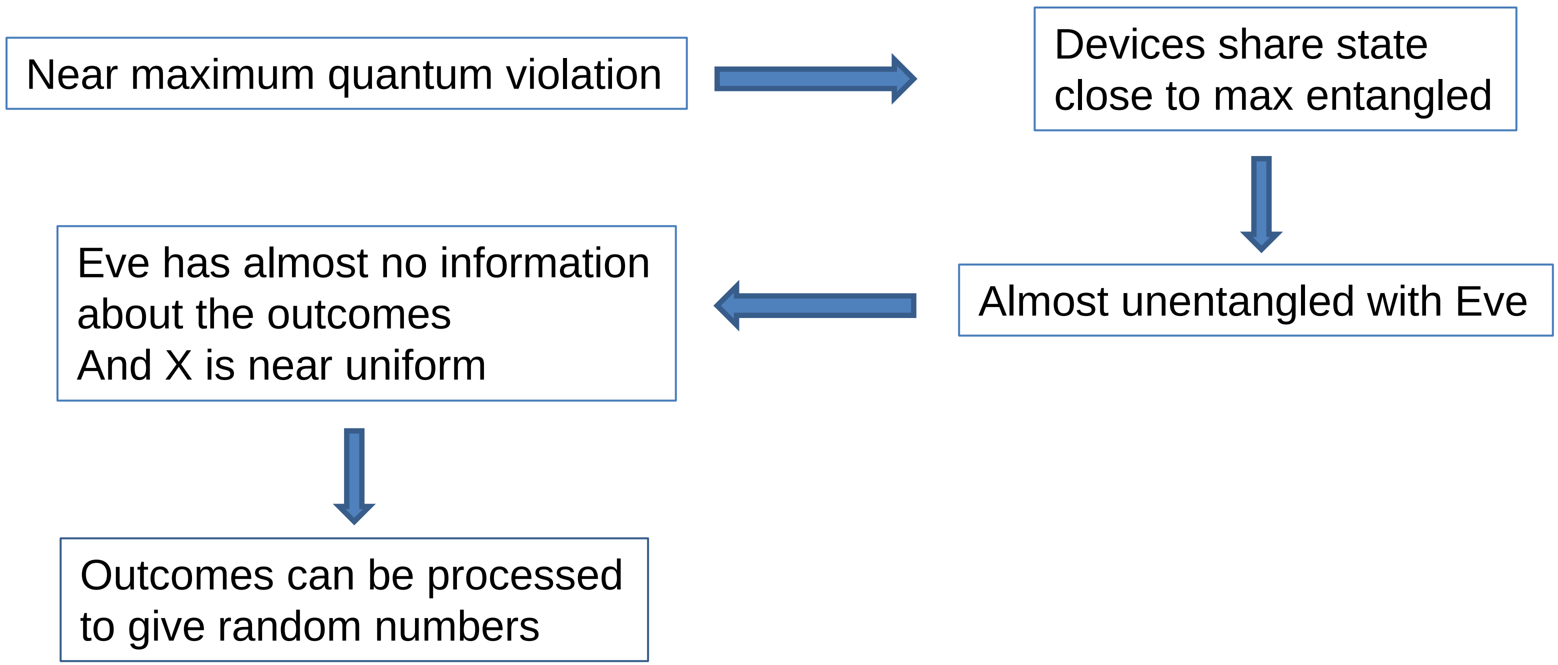
$$|\psi\rangle_{AB} \otimes |\phi\rangle_E$$

← Eve has no information about the outcomes
And X is uniform

↓

Outcomes can be used as random numbers

Device-independent randomness expansion: main ideas



Device-independent randomness expansion: main ideas

Near maximum quantum violation

↓

Eve has almost no information about the outcomes
And X is near uniform

↓

Outcomes can be processed to give random numbers

Connecting Bell violation with Eve's knowledge

$P_{XY\|AB}$	B	1		3	
	Y	0	1	0	1
A	X				
0	0	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$
	1	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
2	0	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	1	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$

$$P_{\text{win}} = 1 - 2\varepsilon$$

How much can Eve know about X?

Connecting Bell violation with Eve's knowledge

$P_{XY\|AB}$	B	1		3	
	Y	0	1	0	1
A	X				
0	0	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$
	1	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
2	0	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	1	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$

$$P_{\text{win}} = 1 - 2\varepsilon$$

How much can Eve know about X?

$$P_{XY|AB} = \sum_z p_z P_{XY|ABz}$$

Convex combination

Quantum-realizable distributions

Connecting Bell violation with Eve's knowledge

$P_{XY\|AB}$	B	1		3	
	Y	0	1	0	1
A	X				
0	0	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$
	1	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
2	0	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	1	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$

$$P_{\text{win}} = 1 - 2\varepsilon$$

How much can Eve know about X?

$$P_{XY|AB} = \sum_z p_z P_{XY|ABz}$$

Convex combination

Any non-signalling distribution

Connecting Bell violation with Eve's knowledge

$P_{XY\|AB}$	B	1		3	
	Y	0	1	0	1
A	X				
0	0	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$
	1	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
2	0	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	1	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$

$$P_{\text{win}} = 1 - 2\varepsilon$$

How much can Eve know about X?

$$P_{XY|AB} = \sum_z p_z P_{XY|ABz}$$

Convex combination

Any non-signalling distribution

$$P_{XY|AB} = (1-4\varepsilon)\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \varepsilon\left(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right)$$

Eve has no knowledge about X

Eve knows X perfectly

Connecting Bell violation with Eve's knowledge

$P_{XY\|AB}$		$B=1$, $Y=0$	$B=1$, $Y=1$	$B=3$, $Y=0$	$B=3$, $Y=1$
A	X				
0	0	$\frac{1}{2}-\varepsilon$	ε	ε	$\frac{1}{2}-\varepsilon$
	1	ε	$\frac{1}{2}-\varepsilon$	$\frac{1}{2}-\varepsilon$	ε
2	0	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$	ε
	1	ε	$\frac{1}{2}-\varepsilon$	ε	$\frac{1}{2}-\varepsilon$

$$P_{\text{win}} = 1 - 2\varepsilon$$

How much can Eve know about X?

$$P_{XY|AB} = \sum_z p_z P_{XY|ABz}$$

Convex combination

Any non-signalling distribution

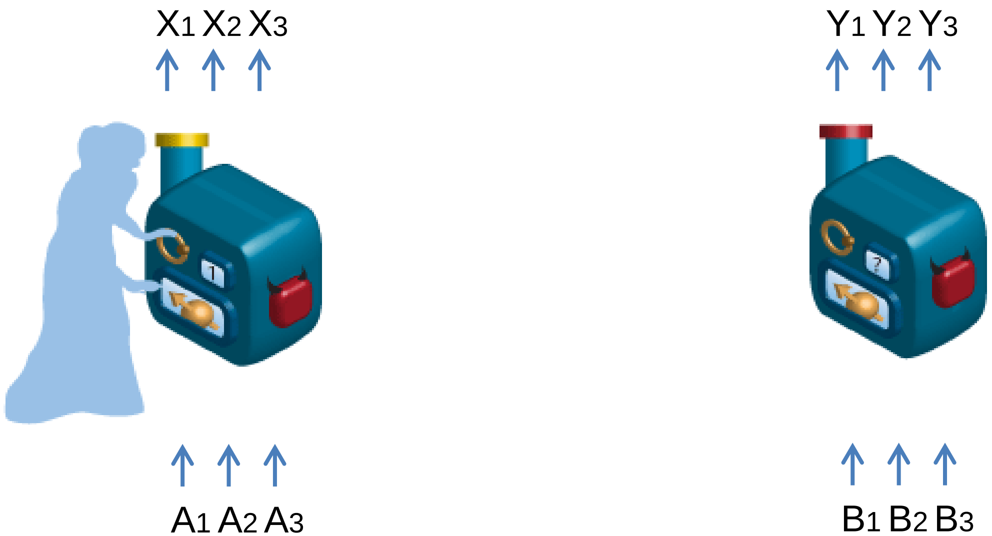
$$P_{XY|AB} = (1-4\varepsilon)\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + \varepsilon\left(\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}\right)$$

Eve has no knowledge about X

Eve knows X perfectly

Non-signalling Eve can guess X with probability $4\varepsilon + \frac{1}{2}(1-4\varepsilon) = \frac{1}{2} + 2\varepsilon$

Device-independent randomness expansion protocol: Main ideas



- Doing CHSH test costs randomness
- We want expansion

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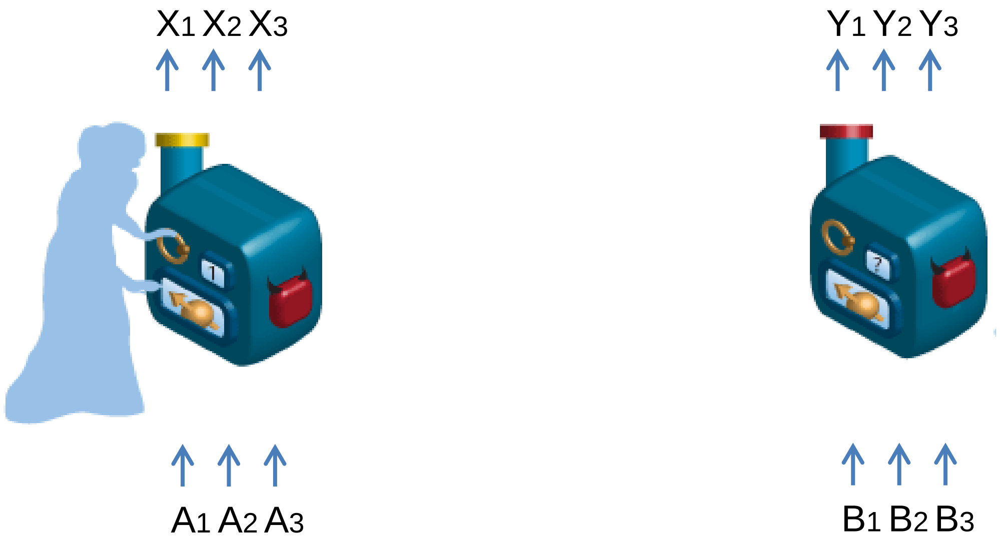
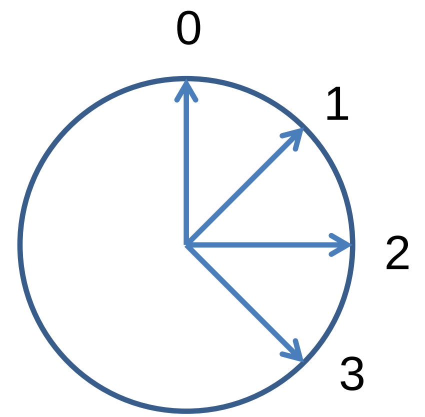
Device-independent randomness expansion protocol: Main ideas



- Divide rounds into "test rounds" (T) and "generation rounds" (G)
- Test rounds are a small subset that cost randomness
- On the generation rounds, fixed inputs are used (no cost), e.g., (try to) measure in $\{|0\rangle, |1\rangle\}$ basis on both

Protocol structure

	A	X		B	Y
G	0	1		0	1
T	2	0		1	1
G	0	1		0	1
T	0	0		1	0
T	2	0		3	0
G	0	1		0	1
G	0	0		0	1
G	0	1		0	0
G	0	1		0	1
G	0	0		0	0
T	0	1		3	0



Use T rounds to check CHSH wins and error rate. For these

If $(\mathrm{A}, \mathrm{B}) = (0,1), (2,1)$ or $(2,3)$, want $X = Y$

If $(A, B) = (0,3)$ want $X \neq Y$

Error rate too high → abort

Protocol structure

	A	X		B	Y
G	0	1		0	1
T	2	0		1	1
G	0	1		0	1
T	0	0		1	0
T	2	0		3	0
G	0	1		0	1
G	0	0		0	1
G	0	1		0	0
G	0	1		0	1
G	0	0		0	0
T	0	1		3	0

Raw string is processed to give final random string

$S_A = 1110110\ldots$

Randomness extraction

$01101\ldots$

NB: randomness extraction needs a short random seed.

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Proof ingredients

- Protocol acts like a filter: for a significant probability of not aborting, the devices must have a large Bell inequality violation almost every time.

- Large Bell inequality violations implies difficulty for Eve to guess.

- If Eve cannot guess the output well, then we can compress the string to one she cannot guess at all. [via randomness extractor]

Randomness accounting

- Randomness input:
  - To choose the test rounds
  - To choose the tests (2 bits per test)
  - To seed the randomness extractor
- Randomness output:
  - If all goes well about 1 bit per round
- Few test rounds, short seed extractors → expansion

Security definition

- What does it mean for a protocol to be secure?
- Define ideal
- Imagine Alice will randomly decide either to perform the real protocol or the ideal.
- The real protocol is secure if it is virtually impossible to distinguish the two.

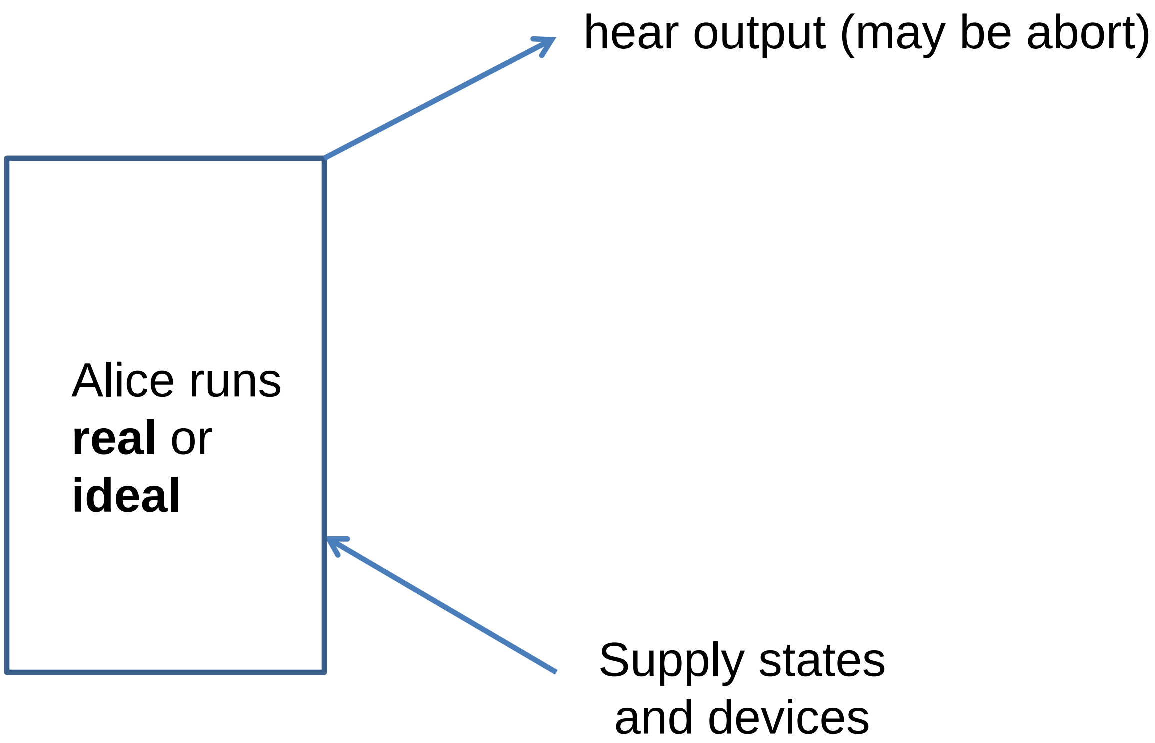
Composable security

- Larger protocol
  - 1.
  - 2.
  - …
  - n. Call randomness expansion sub-protocol
  - n+1.
  - …

Either use **Real** expansion sub-protocol, or **Ideal**

How well can we tell the difference?

Security definition



The ideal

- We want the final state to have the form

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The ideal

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$$\tilde{\rho}_{AE} = \sum_{x} \frac{1}{|X|} |x\rangle\langle x|_A \otimes \rho_E$$

- However, we **don’t** simply define the ideal to output a state of this form.
- (It would be easy to distinguish this from the real protocol, e.g. by forcing real to abort)

The ideal

- Instead, take the ideal protocol to be the real protocol modified such that if it does not abort, right at the end Alice replaces her output by a perfect random string.

$$\sum_x \frac{1}{|X|} |x\rangle\langle x|_A \otimes \rho_E$$

The ideal

- With the ideal defined in this way, it is impossible to distinguish the real and ideal based on abort.

- Only way to distinguish is if both:
  - The protocol does not abort; and
  - The output can be distinguished from a perfect random string

$$D\left(\rho_{AE}, \sum_{x} \frac{1}{|X|} |x\rangle\langle x|_A \otimes \rho_E\right) > 0$$

real

The ideal

- Thus, the security statement is a bound on the *a priori* probability that the protocol does not abort and the output can be distinguished from perfect randomness over all possible devices.

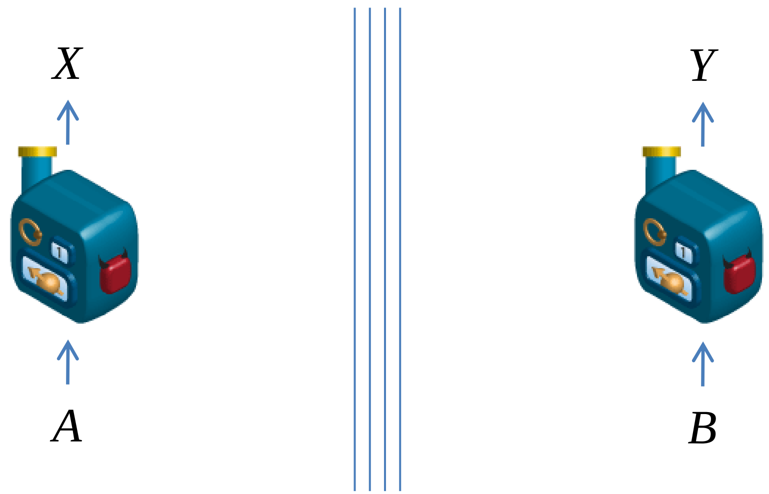
- NB: we don't make statements of the form "Given the protocol did not abort, the output is secure (except with very small probability)"

Technological aspects

- We have theoretical proofs: what about in practice?

Technological aspects

- What about in practice?
- Key ingredient is a Bell inequality violation
  - Need to close detection loophole

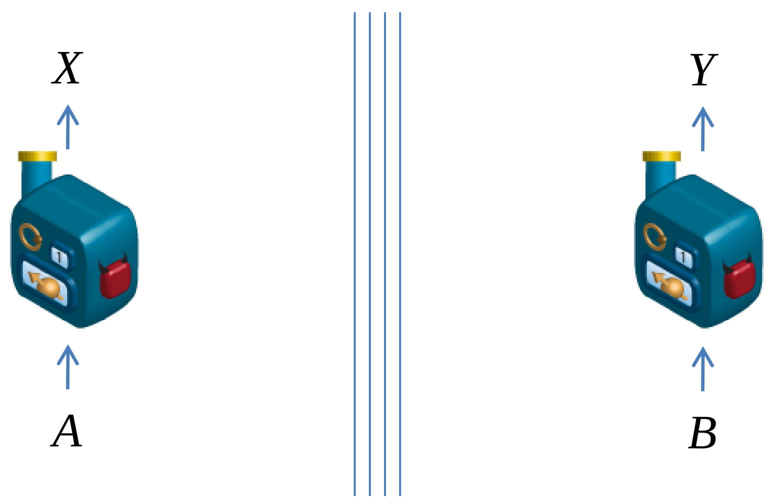


$P_{XY|AB}$ must violate a Bell inequality
In order to verify this, have to include failure to detect events

Technological aspects

- What about in practice?
- Key ingredient is a Bell inequality violation
  - Need to close detection loophole

  NB: easier to do this than for QKD



$P_{XY|AB}$ must violate a Bell inequality
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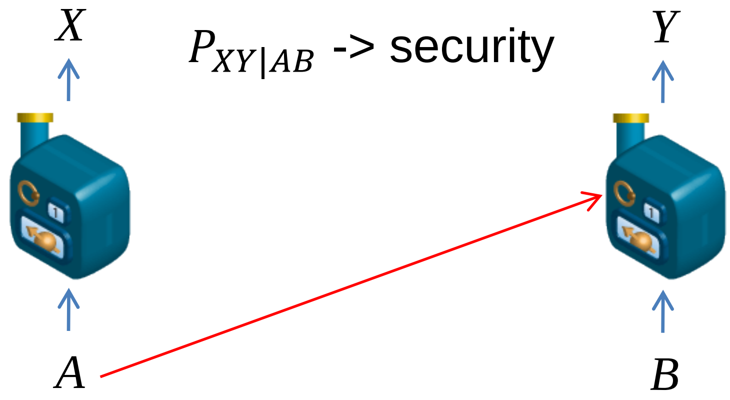
Technological aspects

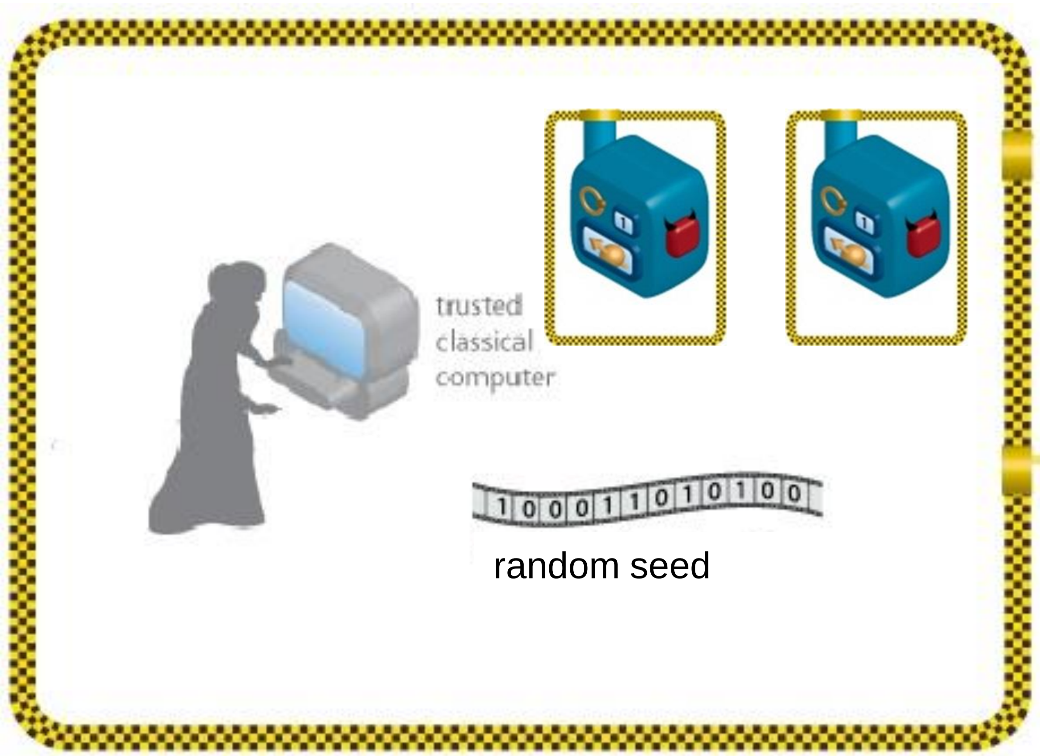
- What about in practice?
  - Need to close detection loophole
  - (Note: no need to close locality loophole; although it doesn't hurt)



Technological aspects

- What about in practice?
  - Need to close detection loophole
  - (Note: no need to close locality loophole; although it doesn't hurt)
  - Need them to be faster to compete with current approaches

Some references



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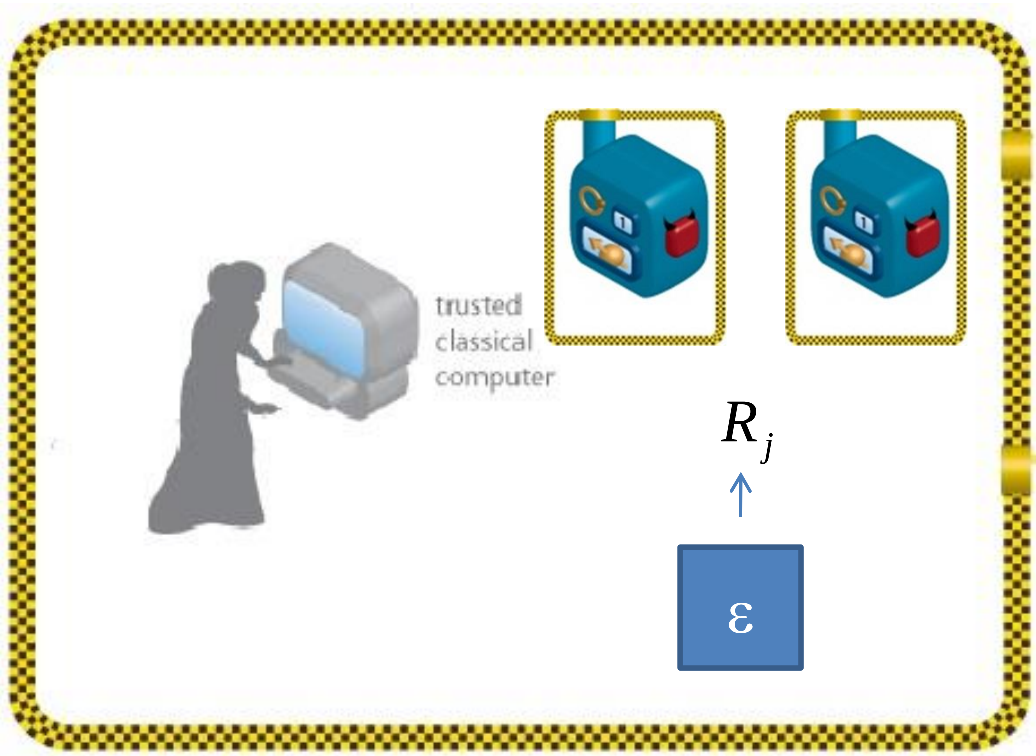
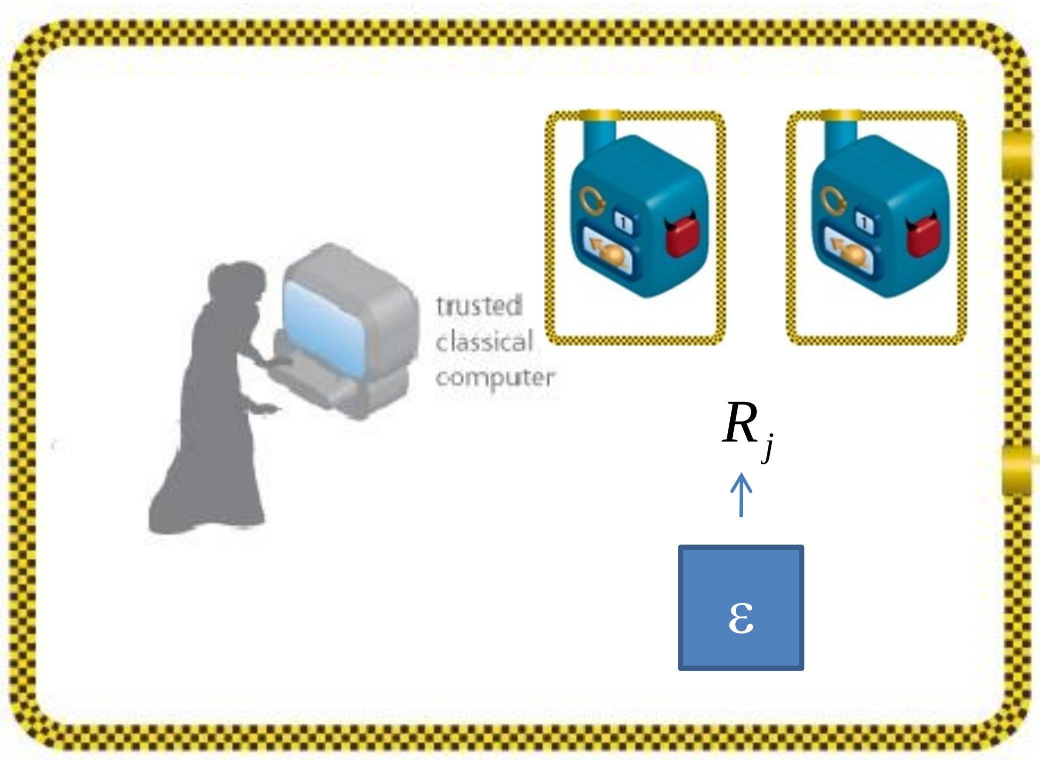
Related task: Randomness Amplification



Imperfect randomness:
- Looks random to Alice
- Partly correlated with other information (that may be held by Eve)

Want to generate perfect randomness

Related task: Randomness Amplification



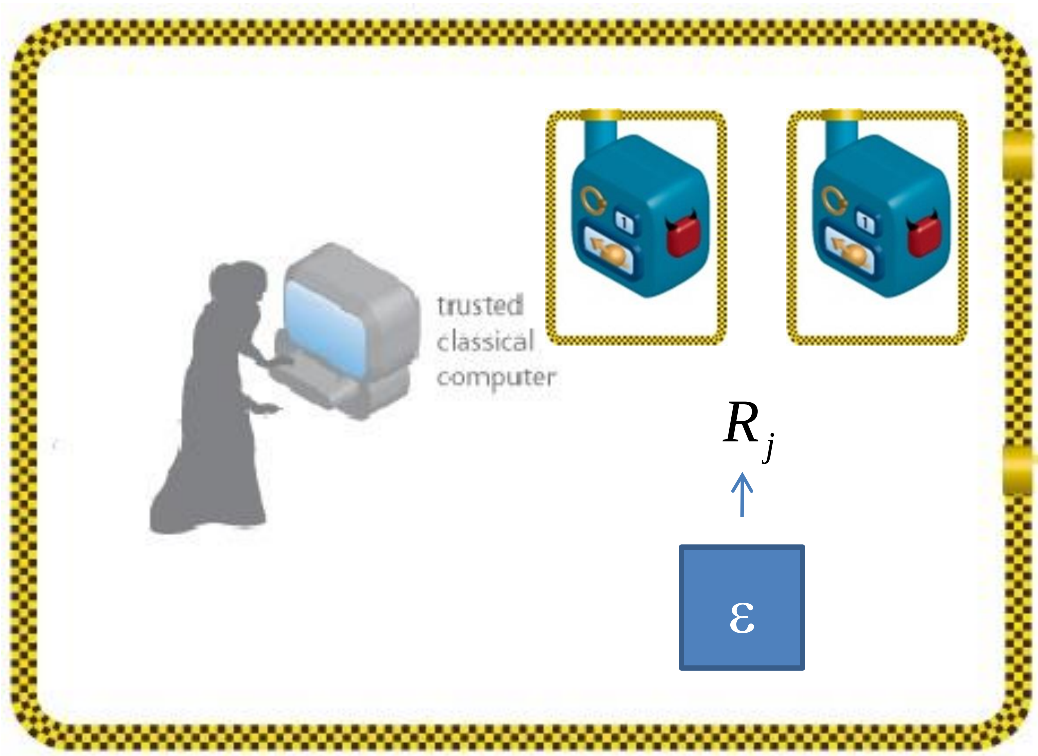
Imperfect randomness:
- Looks random to Alice
- Partly correlated with other information (that may be held by Eve)

E.g., Santha-Vazirani source [FOCS 84]
Limitation to the bias of each bit conditioned on previous ones and adversary.

$$P_{R_j|W} \in \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right]$$

Want to generate perfect randomness

Related task: Randomness Amplification



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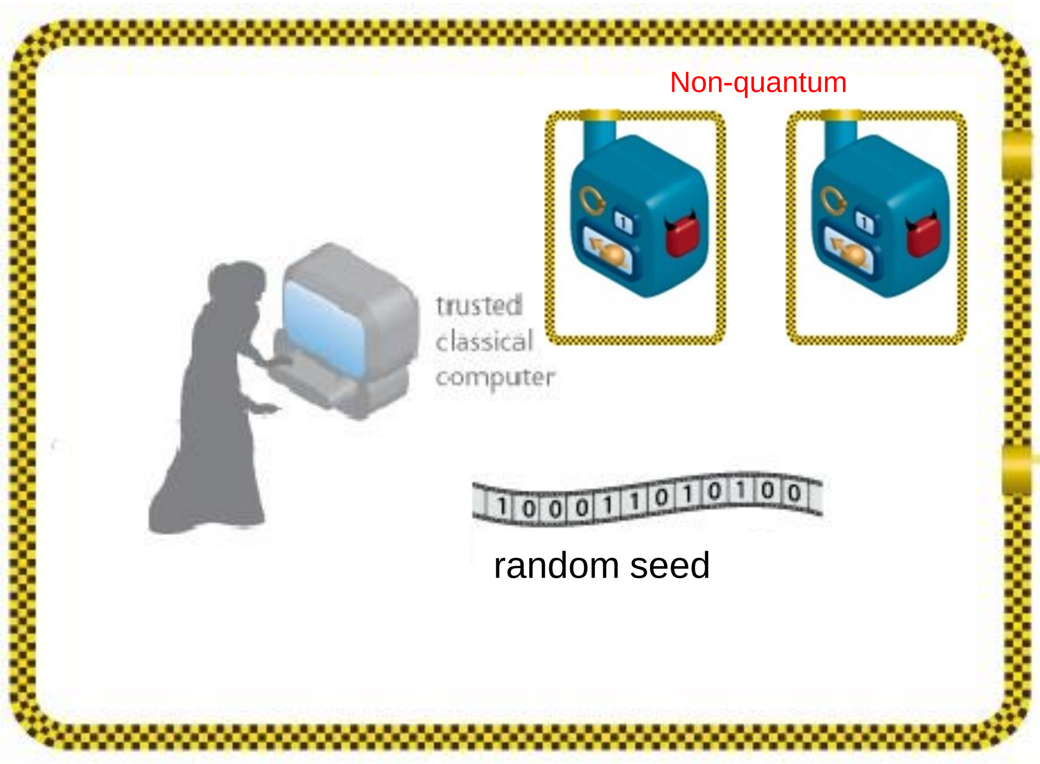
Want to generate perfect randomness

Another interesting scenario



Randomness expansion against non-signalling eavesdroppers

Summary

- Classical protocols aim to provide time-limited security

- Standard quantum protocols allow this to be upgraded to unconditional security

- Device-independent protocols allow security against device failure or tampering

more security

fewer assumptions

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Summary

- Advantages:
  - **weaker assumptions -> more security**
  - certify security on-the-fly (calibration errors automatically caught).
- Open challenges
  - Increased speed
  - Sensible ways to reuse untrusted devices
  - Can we get security against no-signalling adversaries?