# Converse bounds for private communication over quantum channels

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Based on arXiv:1602.08898 [WTB16]

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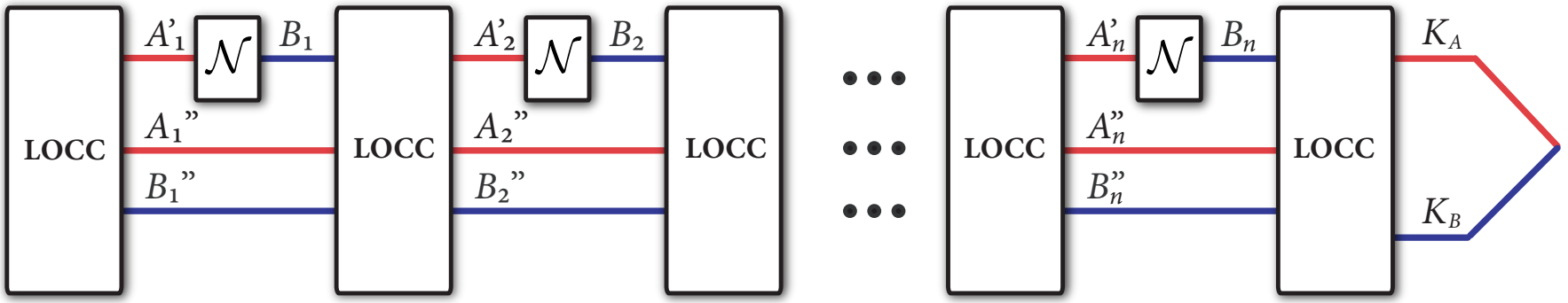
# Setting

Given is a quantum channel $\mathcal{N}$ and a QKD protocol that uses it $n$ times:




**Non-asymptotic private capacity**: Maximum rate of $\varepsilon$-close secret key achievable using channel $n$ times and $N_S$ mean photons per channel use:

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, N_S, \varepsilon) \equiv \sup\{P : (n, P, N_S, \varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\}.$$

If no photon number constraint, then consider

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, \varepsilon) = \sup_{N_S \geq 0} \hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, N_S, \varepsilon).$$

# Main question

- Practical question: How to characterize $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, N_S, \varepsilon)$ for all $n \geq 1$, $N_S \geq 0$, and $\varepsilon \in (0, 1)$?
- How to characterize $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, \varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0, 1)$?
- The answers give the fundamental limitations of QKD.
- Upper bounds on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, N_S, \varepsilon)$ and $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, \varepsilon)$ can be used as benchmarks for quantum repeaters [TGW14].
- This talk discusses the tightest known upper bound on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n, \varepsilon)$ for channels of practical interest and thus represents the best known benchmark for quantum repeaters [WTB16].

# What was known before?

- Begin by reviewing what is known
- Let's leap back to QCrypt 2014:


- Takeoka presented results of [TGW14].

# [TGW14] bound with energy constraint

- Most interested in the photon loss channel:

$$\mathcal{L}_\eta : \quad \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e}$$

  where transmissivity $\eta \in [0, 1]$ and environment in vacuum state.

- Practical question is tough, so consider limiting cases...

- [TGW14] bound: Consider the limit as $n \to \infty$ and then $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n, N_S, \varepsilon) \leq g((1+\eta)N_S/2) - g((1-\eta)N_S/2)$$

$$\text{where} \quad g(x) \equiv (x+1)\log_2(x+1) - x \log_2 x$$

  is entropy of bosonic thermal state with mean photon number $x$.

- Based on the squashed entanglement measure [CW04].

# [TGW14] bound without energy constraint

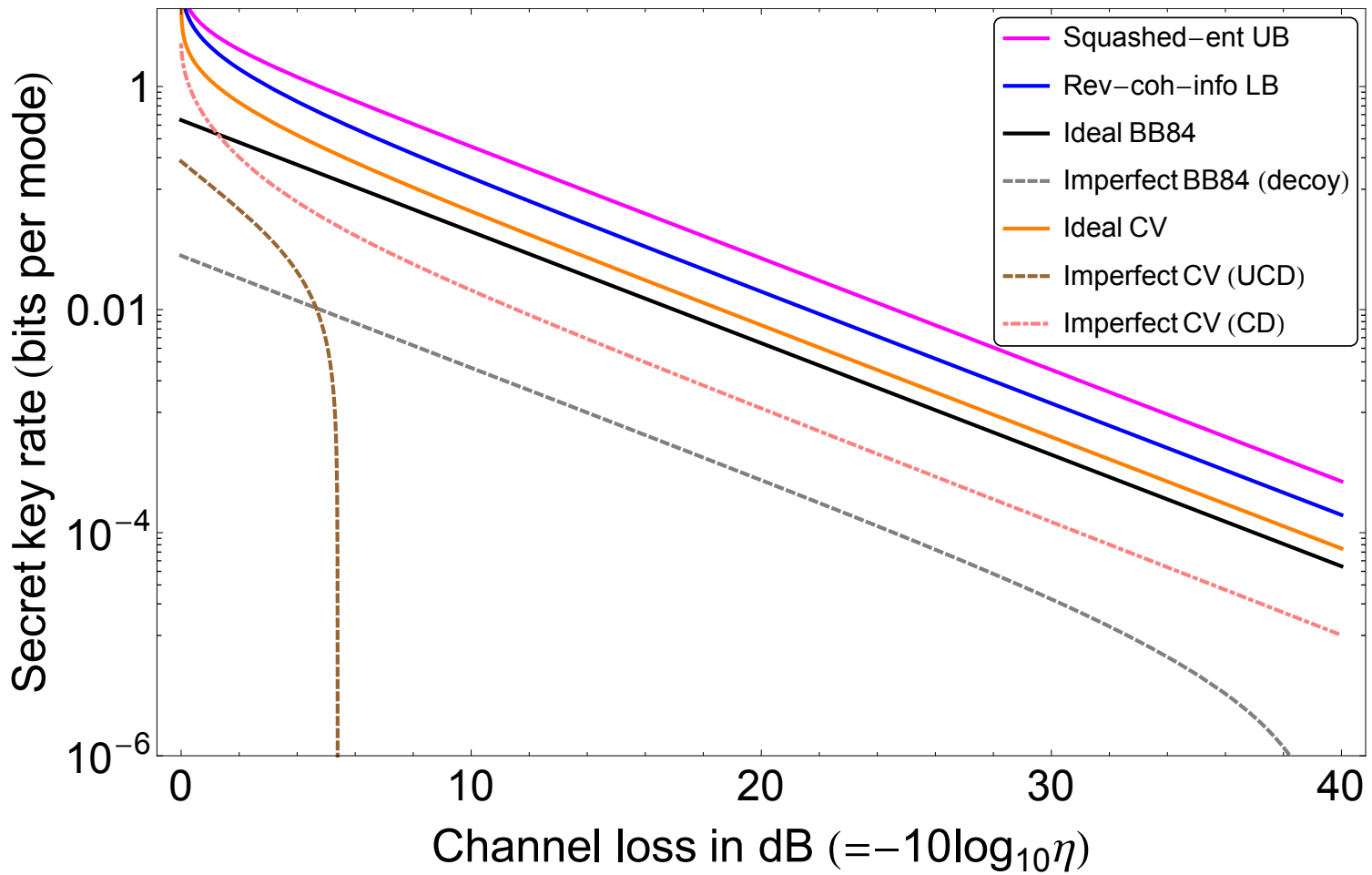
- Optimizing over energy gives the unconstrained [TGW14] bound:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n, \varepsilon) \leq \log_2 \left( \frac{1+\eta}{1-\eta} \right).$$

  essentially because $\sup_{N_S \geq 0} g((1+\eta)N_S/2) - g((1-\eta)N_S/2) = \log_2 \left( \frac{1+\eta}{1-\eta} \right)$.

- [TGW14] established existence of a fundamental rate-loss trade-off for any possible QKD protocol that uses a photon-loss channel.
- Bound is finite for all $\eta \in [0, 1)$ and depends only on $\eta$.
- Main drawback is that it is an asymptotic statement and thus has limited applicability in practice.
- (Original proof didn't address issue with unbounded shield systems — now fixed in [W16])

# Fundamental rate-loss trade-off from [TGW14]




Can translate $x$-axis to km by assuming fiber has 0.2 dB loss / km

# [PLOB15] bound

- By a different method, [PLOB15] established the upper bound:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n, \varepsilon) \leq \log_2 \left( \frac{1}{1-\eta} \right). \qquad (\star)$$

- In fact, with an infinite number of channel uses, infinite energy, and perfect quantum computers for Alice and Bob, the bound is tight:

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n, \varepsilon) = \log_2 \left( \frac{1}{1-\eta} \right).$$

- Drawbacks are the same: An asymptotic statement, and thus says little for practical protocols (called a weak converse bound)
- Method used in [PLOB15] does not give any improved bound for protocols using finite energy (Finite-energy SE can be tighter [GEW16])
- (Proof of $(\star)$ in [PLOB15, Supp. Mat., Sec. III] does not address issue of unbounded shield systems, & thus their proof gives trivial upper bound of $\infty$ for LHS of $(\star)$ — this issue is addressed and fixed in [WTB16])

# Upper bound for non-asymptotic private capacity [WTB16]

## Bound on Non-Asymptotic Private Capacity

One consequence of the meta-converse approach in [WTB16]:

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log_2\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n},$$

where $C(\varepsilon) \equiv \log_2 6 + 2\log_2\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

- Can be used to assess the performance of any practical quantum repeater which uses a loss channel $n$ times for desired security $\varepsilon$.
- Other variations of this bound are possible if $\eta$ is not the same for each channel use, if $\eta$ is chosen adversarially, etc.
- Remaining technical questions: Improve $C(\varepsilon)$ to $\log_2\left(\frac{1}{1-\varepsilon}\right)$? Finite-energy bound?

# Meta-converse approach from [WTB16]

- Building on [Bla74, BDSW96, BK98, HW01, HHHO05, Che05, DJKR06, HHHO09, CKR09, PPV10, BD11, Li14, TH13, TT15, MLDS$^+$13, WWY14, TWW14, DPR15, TBR15, PLOB15]
- Meta-converse approach starts by using hypothesis testing relative entropy to compare the actual state resulting from the protocol to a separable state, the latter being useless for private comm.
- The approach extracts the relevant parameters of the protocol ($n$, rate $P$, and $\varepsilon$) and relates them via an information-like quantity.
- The meta-converse leads to various other bounds, including Renyi-entropic strong converse bounds and others in terms of relative entropy and relative entropy variance.
- Result: We get the tightest known upper bounds for non-asymptotic private capacity of many channels of practical interest.

# Information measures

- Hypothesis testing relative entropy defined for a state $\rho$, positive semi-definite operator $\sigma$, and $\varepsilon \in [0,1]$ as

$$D_H^\varepsilon(\rho\|\sigma) \equiv -\log\left[\min\{\mathrm{Tr}\{\Lambda\sigma\} : 0 \leq \Lambda \leq I \wedge \mathrm{Tr}\{\Lambda\rho\} \geq 1-\varepsilon\}\right].$$

- Has a second-order expansion for i.i.d. states:

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n).$$

where
$$D(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log\rho - \log\sigma]\},$$
$$V(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log\rho - \log\sigma - D(\rho\|\sigma)]^2\}$$
$$\Phi(a) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} dx\ \exp\left(-x^2/2\right)$$

# Example: Dephasing channel [TBR15, WTB16]

For the qubit dephasing channel

$$\mathcal{Z}^{\gamma} : \rho \mapsto (1-\gamma)\,\rho + \gamma Z \rho Z,$$

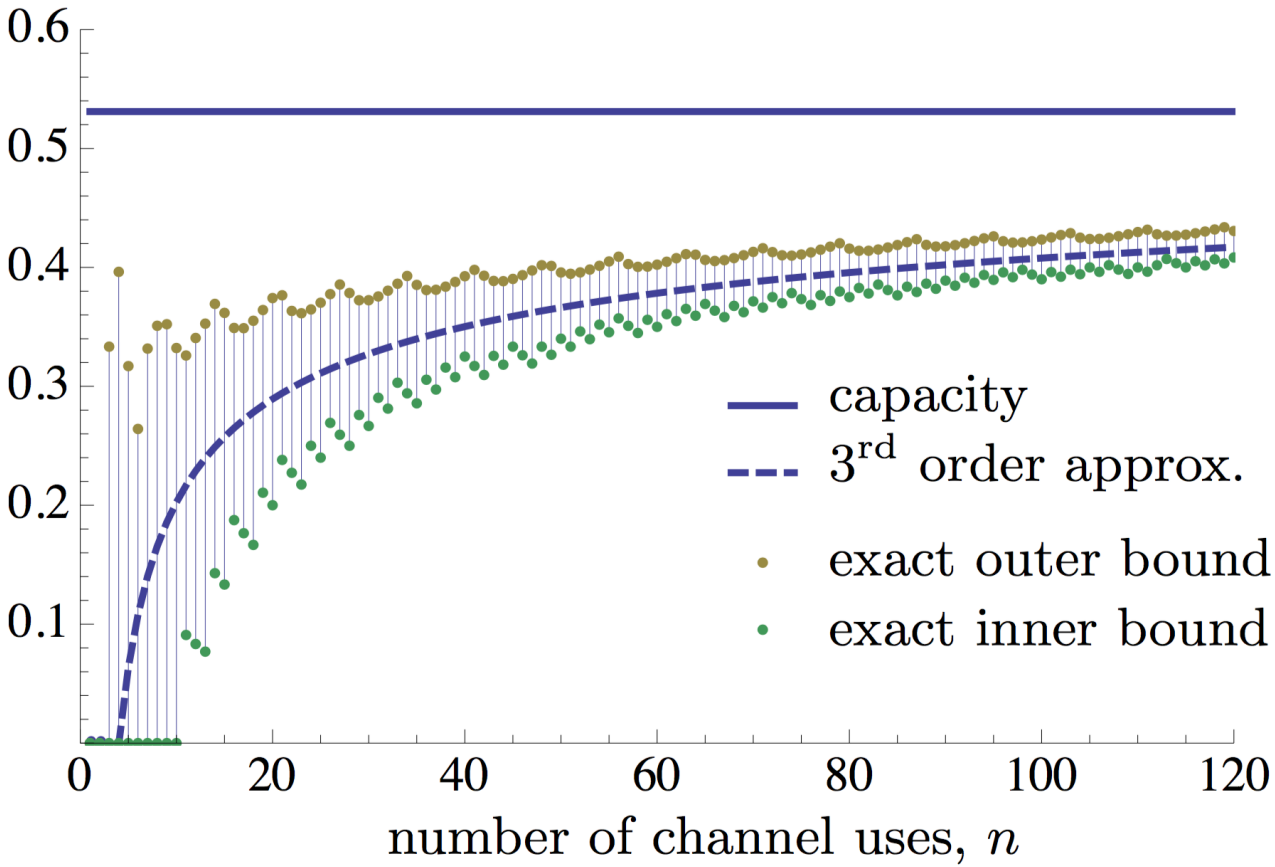
with $\gamma \in (0,1)$, the non-asymptotic private capacity $\hat{P}^{\leftrightarrow}(n,\varepsilon)$ satisfies

$$\hat{P}^{\leftrightarrow}(n,\varepsilon) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}}\,\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right),$$

where $\Phi$ is the cumulative standard Gaussian distribution, $h(\gamma)$ denotes the binary entropy and $v(\gamma)$ the corresponding variance, defined as

$$h(\gamma) \equiv -\gamma \log \gamma - (1-\gamma)\log(1-\gamma),$$
$$v(\gamma) \equiv \gamma(\log\gamma + h(\gamma))^2 + (1-\gamma)(\log(1-\gamma) + h(\gamma))^2.$$

# Example: Dephasing channel [TBR15, WTB16]




(c) Comparison of strict bounds with third order approximation for $\varepsilon = 5\%$.

($\gamma = 0.1$, plot taken from [TBR15])

# Example: Erasure channel [TBR15, WTB16]

For the qubit erasure channel

$$\mathcal{E}^{p}_{A'\to B} : \rho_{A'} \mapsto (1-p)\rho_B + p|e\rangle\langle e|_B$$

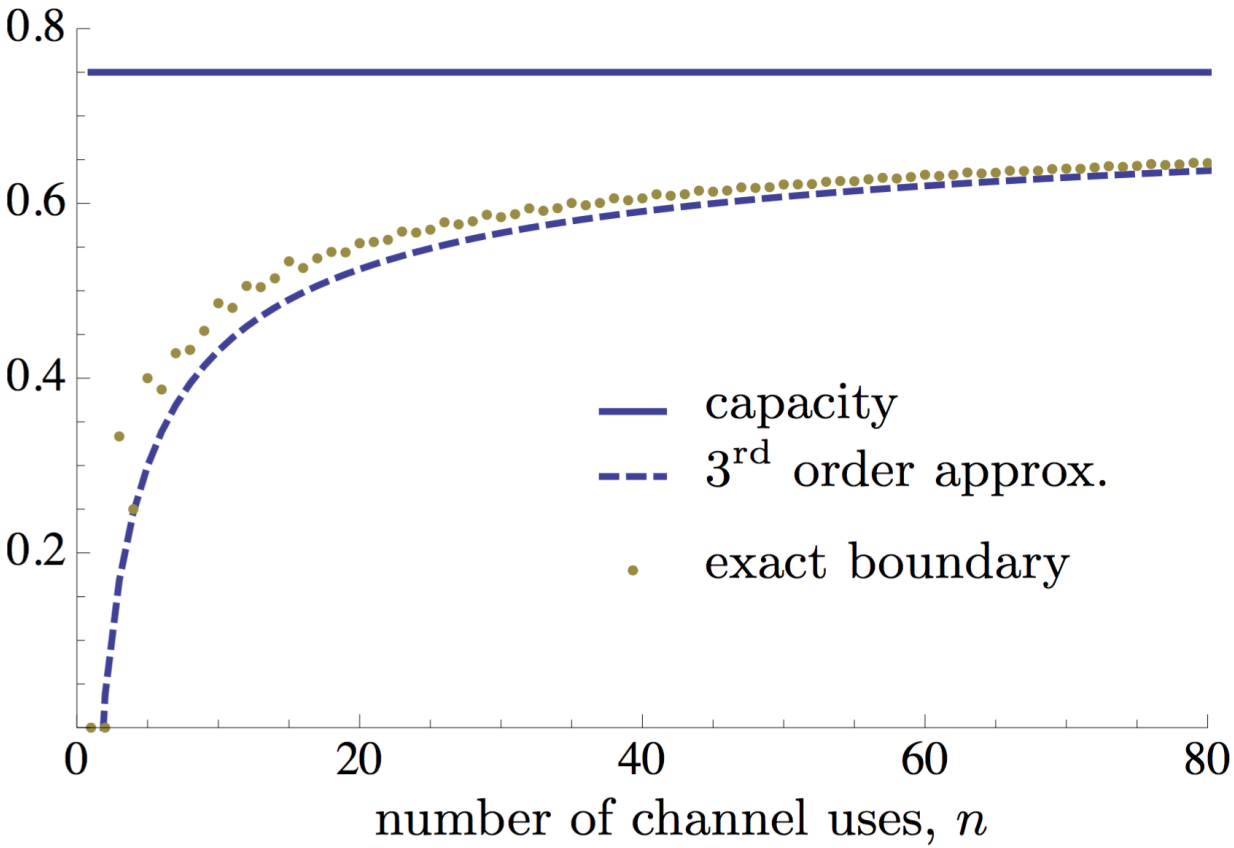
with $p \in (0,1)$, the non-asymptotic private capacity $\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon)$ satisfies

$$\varepsilon = \sum_{l=n-k+1}^{n} \binom{n}{l} p^l (1-p)^{n-l} \left(1 - 2^{n\left(1-\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon)\right)-l}\right) .$$

Moreover, the following expansion holds

$$\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon) = 1 - p + \sqrt{\frac{p(1-p)}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{1}{n}\right) .$$

# Example: Erasure channel [TBR15, WTB16]




(b) Comparison of exact bounds with third order approximation.

($p = 0.25$, plot taken from [TBR15])

# Second-order expansions of converse bounds [WTB16]

## Theorem

*If a finite-dim. quantum channel* $\mathcal{N}_{A'\to B}$ *is covariant, then*

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon) \leq E_R(A;B)_\rho + \sqrt{\frac{V^{\varepsilon}_{E_R}(A;B)_\rho}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right),$$

*where* $\rho_{AB} = \mathcal{N}_{A'\to B}(\Phi_{AA'})$, $E_R(A;B)_\rho$ *is the relative entropy of entanglement,*

$$\text{and} \qquad V^{\varepsilon}_{E_R}(A;B)_\rho \equiv \begin{cases} \max_{\sigma_{AB'}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon < 1/2 \\ \min_{\sigma_{AB}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon \geq 1/2 \end{cases},$$

*with* $\Pi_{\mathcal{S}} \subseteq \mathcal{S}(A\!:\!B)$ *the set of states achieving minimum in* $E_R(A;B)_\rho$

# Application: Quantum Gaussian channels [WTB16]

## Definitions of quantum Gaussian channels

$$\text{Thermal channel } \mathcal{L}_{\eta,N_B}: \quad \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e},$$
$$\text{Amplifier channel } \mathcal{A}_{G,N_B}: \quad \hat{b} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^{\dagger},$$
$$\text{Additive-noise channel } \mathcal{W}_{\xi}: \quad \hat{b} = \hat{a} + (x+ip)/\sqrt{2},$$

- Thermal channel has transmissivity $\eta \in [0,1]$ and environment prepared in thermal state of mean photon number $N_B$.
- Amplifier channel has gain $G \in [1,\infty)$ and environment prepared in thermal state of mean photon number $N_B$.
- If $N_B = 0$, then channels are quantum-limited.
- Additive noise channel has $x$ and $p$ be zero-mean Gaussian random variables with variance $\xi \geq 0$.

# Unconstrained rel. entropies of entanglement [PLOB15]

- For the thermal channel $\mathcal{L}_{\eta,N_B}$, $E_R$ evaluates to

$$-\log_2\left((1-\eta)\,\eta^{N_B}\right) - g(N_B).$$

- For the amplifier channel $\mathcal{A}_{G,N_B}$, $E_R$ evaluates to

$$\log_2\left(\frac{G^{N_B+1}}{G-1}\right) - g(N_B).$$

- For the additive noise channel $\mathcal{W}_\xi$, $E_R$ evaluates to

$$\frac{\xi-1}{\ln 2} - \log_2 \xi.$$

# Unconstrained relative entropy variances [WTB16]

Let $V_{\mathcal{L}_{\eta,N_B}}$, $V_{\mathcal{A}_{G,N_B}}$, and $V_{\mathcal{W}_\xi}$ be the unconstrained relative entropy variances of the thermalizing, amplifier, and additive-noise channels, respectively:

$$
\begin{aligned}
V_{\mathcal{L}_{\eta,N_B}} &\equiv N_B(N_B+1)\log_2^2(\eta\left[N_B+1\right]/N_B),\\
V_{\mathcal{A}_{G,N_B}} &\equiv N_B(N_B+1)\log_2^2(G^{-1}\left[N_B+1\right]/N_B),\\
V_{\mathcal{W}_\xi} &\equiv (1-\xi)^2/\ln^2 2.
\end{aligned}
$$

Can compute these from a general formula for relative entropy variance of two Gaussian states [WTLB16].

# Strong converse bounds for Gaussian channels [WTB16]

## Theorem

*The following strong converse bounds hold for* $\varepsilon \in (0,1)$*:*

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_{\eta,N_B}}(n,\varepsilon) \leq -\log_2\left((1-\eta)\,\eta^{N_B}\right) - g(N_B) + \sqrt{\frac{2V_{\mathcal{L}_{\eta,N_B}}}{n(1-\varepsilon)}} + C(\varepsilon)/n,$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{A}_{G,N_B}}(n,\varepsilon) \leq \log_2\left(\frac{G^{N_B+1}}{G-1}\right) - g(N_B) + \sqrt{\frac{2V_{\mathcal{A}_{G,N_B}}}{n(1-\varepsilon)}} + C(\varepsilon)/n,$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{W}_\xi}(n,\varepsilon) \leq \frac{\xi-1}{\ln 2} - \log_2\xi + \sqrt{\frac{2V_{\mathcal{W}_\xi}}{n(1-\varepsilon)}} + C(\varepsilon)/n.$$

# Strong converse for quantum-limited channels [WTB16]

## Corollary

*For the pure-loss channel $\mathcal{L}_\eta$ and quantum-limited amplifier channel $\mathcal{A}_G$, the following bounds hold*

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log_2\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n},$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{A}_G}(n,\varepsilon) \leq \log\left(\frac{1}{1-1/G}\right) + \frac{C(\varepsilon)}{n}.$$

# Summary

- We have established bounds for QKD protocols conducted over quantum channels that are unassisted by quantum repeaters.
- Meta-converse has several applications, including strong converse bounds and second-order characterizations of private communication
- The bounds are related to the relative entropy of entanglement and sharpen known upper bounds on rates of QKD protocols
- We establish the strong converse property for the two-way assisted private capacity of the pure-loss and quantum-limited amplifier channels. We also get strong converse rates for other quantum Gaussian channels.
- We have generalized these results to broadcast channels with a single sender and multiple receivers [TSW16]
- Squashed entanglement technique applied more generally in [GEW16]

# Methods

- As said before, we build on a variety of techniques and approaches given in previous literature:
- Meta-converse approach for hypothesis testing [Li14, TH13, DPR15], classical communication [TT15], and quantum communication [TWW14, TBR15]
- Private states [HHHO05, HHHO09], a privacy test [HHH$^+$08b, HHH$^+$08a], and relative entropy of entanglement as an upper bound on distillable key [HHHO05, HHHO09]
- Gaussian states and channels [HW01] and formulas for relative entropy for Gaussian states [Che05, PLOB15]
- $\varepsilon$-relative entropy of entanglement [BD11] and sandwiched Renyi relative entropy [MLDS$^+$13, WWY14]
- Reduction of adaptive protocols to non-adaptive ones via simulation of channels by teleportation [BDSW96, PLOB15]

# Private states

## Tripartite key state

A tripartite key state $\gamma_{ABE}$ contains $\log K$ bits of secret key if there exists a state $\sigma_E$ and measurement channels $\mathcal{M}_A$ and $\mathcal{M}_B$ such that

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E.$$

## Bipartite private state

A bipartite private state $\gamma_{ABA'B'}$ has the following form:

$$\gamma_{ABA'B'} = U_{ABA'B'}(\Phi_{AB} \otimes \theta_{A'B'})U^\dagger_{ABA'B'},$$

where $U_{ABA'B'}$ is a "twisting" unitary of the form $U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}$, with each $U^{ij}_{A'B'}$ a unitary, and $\theta_{A'B'}$ a state.

# Private states

- The systems $A'$ and $B'$ are called the "shield" systems because they, along with the twisting unitary, can help to protect the key in systems $AB$ from any party possessing a purification of $\gamma_{ABA'B'}$.

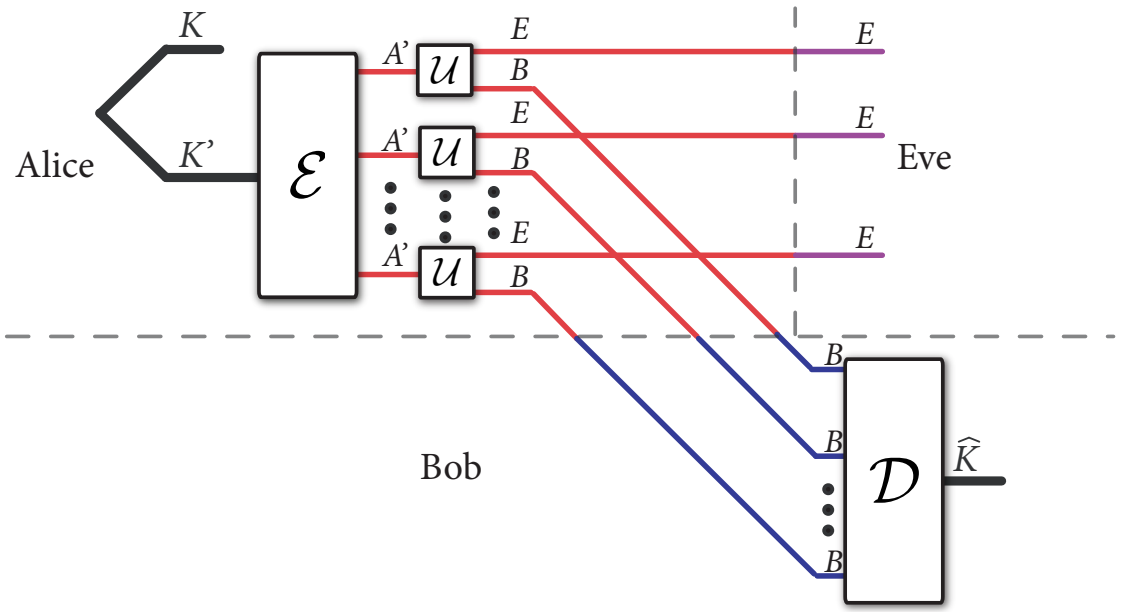
- Such bipartite private states are in one-to-one correspondence with tripartite key states. That is, for every tripartite key state $\gamma_{ABE}$, we can find a bipartite private state and vice versa.

- This correspondence takes on a more physical form: any tripartite protocol whose aim it is to extract tripartite key states is in 1-to-1 correspondence with a bipartite protocol whose aim it is to extract bipartite private states.

# Private communication protocols

## Unassisted private communication

- Given is a quantum channel $\mathcal{N}_{A' \to B}$. Let $U^{\mathcal{N}}_{A' \to BE}$ be an isometric extension of $\mathcal{N}_{A' \to B}$.
- A secret-key generation protocol for $n$ channel uses consists of a triple $\{|K|, \mathcal{E}, \mathcal{D}\}$, where $|K|$ is the size of the secret key to be generated, $\mathcal{E}_{K' \to A'^n}$ is the encoder, and $\mathcal{D}_{B^n \to \hat{K}}$ is the decoder.



# Private communication protocols

## Unassisted private communication

- A triple $(n, P, \varepsilon)$ consists of the number $n$ of channel uses, the rate $P$ of secret-key generation, and the error $\varepsilon \in [0, 1]$.
- Such a triple is achievable on $\mathcal{N}_{A' \to B}$ if there exists a secret-key generation protocol $\{|K|, \mathcal{E}, \mathcal{D}\}$ and some state $\omega_{E^n}$ such that $\frac{1}{n} \log |K| \geq P$ and

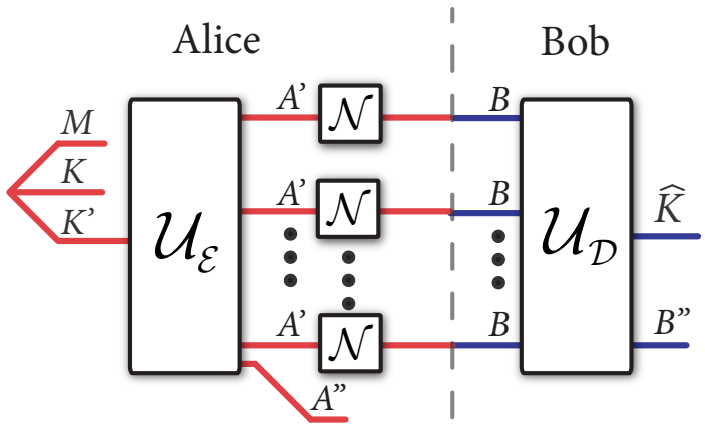
$$F(\overline{\Phi}_{K\hat{K}} \otimes \omega_{E^n}, \rho_{K\hat{K}E^n}) \geq 1 - \varepsilon,$$

where $\rho_{K\hat{K}E^n} \equiv (\mathcal{D}_{B^n \to \hat{K}} \circ (\mathcal{U}^{\mathcal{N}}_{A' \to BE})^{\otimes n} \circ \mathcal{E}_{K' \to A'^n})(\overline{\Phi}_{KK'})$ and

$$\overline{\Phi}_{KK'} \equiv \frac{1}{|K|} \sum_{i=0}^{|K|-1} |i\rangle\langle i|_K \otimes |i\rangle\langle i|_{K'}.$$

# Equivalent bipartite protocol

Can reformulate such a protocol in the bipartite picture: perform every step coherently, with the goal to produce a bipartite private state




Due to equivalence between tripartite and bipartite pictures

$$F(\gamma_{K_A K_B S_A S_B}, \rho_{K\hat{K}MA''B''}) \geq 1 - \varepsilon,$$

for some private state $\gamma_{K_A K_B S_A S_B}$, where we identify $K_A \equiv K$, $K_B \equiv \hat{K}$, $S_A \equiv MA''$, and $S_B \equiv B''$, and

$$\rho_{K\hat{K}MA''B''} \equiv (\mathcal{U}^{\mathcal{D}}_{B^n \to \hat{K}B''} \circ (\mathcal{U}^{\mathcal{N}}_{A' \to BE})^{\otimes n} \circ \mathcal{U}^{\mathcal{E}}_{K' \to A'^n A''})(\Phi^{\mathrm{GHZ}}_{KK'M}).$$

# Non-asymptotic achievable region

## Non-asymptotic fundamental limit

Boundary of the achievable region:

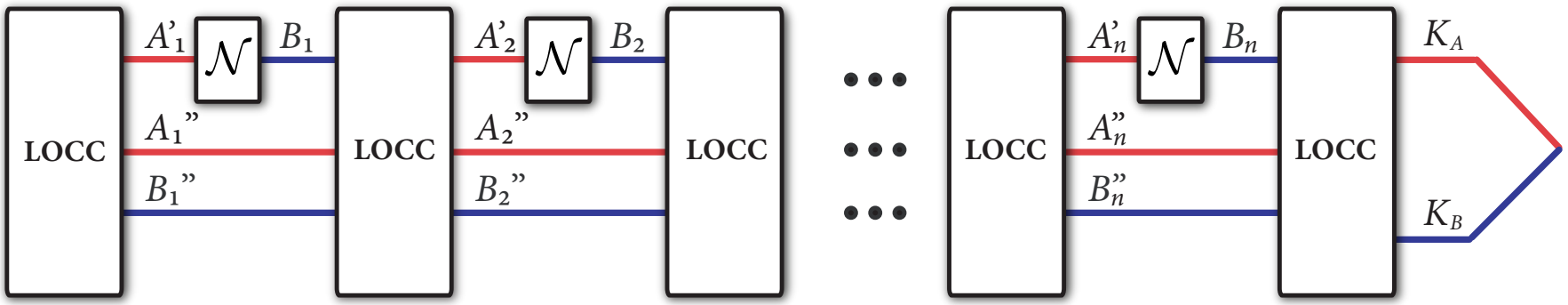
$$\hat{P}_{\mathcal{N}}(n, \varepsilon) \equiv \max \{P : (n, P, \varepsilon) \text{ is achievable for } \mathcal{N}\}, .$$

## Interpretation

- Boundary $\hat{P}_{\mathcal{N}}(n, \varepsilon)$ identifies how rate can change as a function of $n$ for fixed error $\varepsilon$, and 2nd-order coding rates can characterize it

# LOCC-assisted private communication protocols

LOCC-assisted protocols are defined similarly, but allow for rounds of LOCC between channel uses (like in QKD)




Define boundary of non-asymptotic achievable region similarly as

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon) \equiv \max\{P : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\}.$$

# Information measures

- Can use hypothesis testing relative entropy to define the $\varepsilon$-relative entropy of entanglement:

$$E_R^{\varepsilon}(A;B)_{\rho} \equiv \inf_{\sigma_{AB} \in \mathcal{S}(A:B)} D_H^{\varepsilon}(\rho_{AB} \| \sigma_{AB}).$$

  where $\mathcal{S}(A:B)$ is the set of separable states

- Can also define a channel's $\varepsilon$-relative entropy of entanglement:

$$E_R^{\varepsilon}(\mathcal{N}) \equiv \sup_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} E_R^{\varepsilon}(A;B)_{\rho},$$

  where $\rho_{AB} \equiv \mathcal{N}_{A' \to B}(\psi_{AA'})$

- Standard relative entropies of entanglement defined by replacing $D_H^{\varepsilon}$ with quantum relative entropy $D$

# Privacy test

- Can test whether a given state is a $\gamma$-private state by "untwisting" and projecting onto the maximally entangled state:

$$\{\Pi_{ABA'B'}, I_{ABA'B'} - \Pi_{ABA'B'}\},$$

where $\Pi_{ABA'B'} \equiv U_{ABA'B'} (\Phi_{AB} \otimes I_{A'B'}) U^{\dagger}_{ABA'B'}$.

- Let $\varepsilon \in [0, 1]$ and let $\rho_{ABA'B'}$ be an $\varepsilon$-approximate $\gamma$-private state. The probability for $\rho_{ABA'B'}$ to pass the $\gamma$-privacy test satisfies

$$\mathrm{Tr}\{\Pi_{ABA'B'} \rho_{ABA'B'}\} \geq 1 - \varepsilon,$$

- For a separable state $\sigma_{ABA'B'} \in \mathcal{S}(AA' : BB')$, the probability of passing any $\gamma$-privacy test is never larger than $1/K$:

$$\mathrm{Tr}\{\Pi_{ABA'B'} \sigma_{ABA'B'}\} \leq \frac{1}{K},$$

where $K$ is the number of values that the secret key can take.

# Meta-converse bound for private communication

## Theorem

*For any fixed* $\varepsilon \in (0,1)$, *the achievable region satisfies*

$$\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N}).$$

*"One-shot* $\varepsilon$*-private capacity* $\leq$ *channel's* $\varepsilon$*-relative entropy of entanglement."*
*The same bound holds when allowing for a round of LOCC before and after the channel use.*

Proof idea: use monotonicity of $E_R^{\varepsilon}$ with respect to LOCC and use the bounds on the previous slide.

# Meta-converse bound for private communication

## Corollary

*The following bound holds for n channel uses:*

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n} E_R^{\varepsilon}(\mathcal{N}^{\otimes n}).$$

*The same bound holds when allowing for rounds of LOCC before and after all n channel uses. The same bound holds for $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ if the channel $\mathcal{N}$ is teleportation simulable.*

The previous theorem and this corollary then imply all of our previous results, with some extra work needed to establish a formula for the relative entropy variance of Gaussian states.

# Application: Second-order expansions of converse bounds

## Theorem

*If a quantum channel $\mathcal{N}_{A'\to B}$ is teleportation-simulable with associated state $\omega_{AB}$, then*

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon) \leq E_R(A;B)_\omega + \sqrt{\frac{V^{\varepsilon}_{E_R}(A;B)_\omega}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right).$$

*where* $$V^{\varepsilon}_{E_R}(A;B)_\rho \equiv \begin{cases} \max_{\sigma_{AB'}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon < 1/2 \\ \min_{\sigma_{AB}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon \geq 1/2 \end{cases},$$

*with* $\Pi_{\mathcal{S}} \subseteq \mathcal{S}(A\!:\!B)$ *the set of states achieving minimum in* $E_R(A;B)_\rho$

# Tool: Relative entropy variance for Gaussian states

Writing zero-mean Gaussian states in exponential form as

$$\rho = Z_\rho^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\rho \hat{x}\right\}, \qquad \sigma = Z_\sigma^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\sigma \hat{x}\right\},$$

where

$$Z_\rho \equiv \det(V^\rho + i\Omega/2), \qquad Z_\sigma \equiv \det(V^\sigma + i\Omega/2),$$
$$G_\rho \equiv 2i\Omega \operatorname{arcoth}(2V^\rho i\Omega), \qquad G_\sigma \equiv 2i\Omega \operatorname{arcoth}(2V^\sigma i\Omega),$$

and $V^\rho$ and $V^\sigma$ are Wigner function covariance matrices for $\rho$ and $\sigma$.

## Theorem

*For zero-mean Gaussian states $\rho$ and $\sigma$, the relative entropy variance is*

$$V(\rho\|\sigma) = \frac{1}{2}\operatorname{Tr}\{\Delta V^\rho \Delta V^\rho\} + \frac{1}{8}\operatorname{Tr}\{\Delta\Omega\Delta\Omega\},$$

*where $\Delta \equiv G_\rho - G_\sigma$.*

# References I


[BD11] Fernando G. S. L. Brandao and Nilanjana Datta. One-shot rates for entanglement manipulation under non-entangling maps. *IEEE Transactions on Information Theory*, 57(3):1754–1760, March 2011. arXiv:0905.2673.

[BDSW96] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. *Physical Review A*, 54(5):3824–3851, November 1996. arXiv:quant-ph/9604024.

[BK98] Samuel L. Braunstein and H. J. Kimble. Teleportation of continuous quantum variables. *Physical Review Letters*, 80(4):869–872, January 1998.

[Bla74] Richard Blahut. Hypothesis testing and information theory. *IEEE Transactions on Information Theory*, 20(4):405–417, July 1974.

[Che05] Xiao-yu Chen. Gaussian relative entropy of entanglement. *Physical Review A*, 71(6):062320, June 2005. arXiv:quant-ph/0402109.

# References II


[CKR09] Matthias Christandl, Robert König, and Renato Renner. Postselection technique for quantum channels with applications to quantum cryptography. *Physical Review Letters*, 102(2):020504, January 2009. arXiv:0809.3019.

[CW04] Matthias Christandl and Andreas Winter. "Squashed entanglement": An additive entanglement measure. *Journal of Mathematical Physics*, 45(3):829–840, March 2004. arXiv:quant-ph/0308088.

[DJKR06] Igor Devetak, Marius Junge, Christopher King, and Mary Beth Ruskai. Multiplicativity of completely bounded p-norms implies a new additivity result. *Communications in Mathematical Physics*, 266(1):37–63, August 2006. arXiv:quant-ph/0506196.

[DPR15] Nilanjana Datta, Yan Pautrat, and Cambyse Rouzé. Second-order asymptotics for quantum hypothesis testing in settings beyond i.i.d. - quantum lattice systems and more. *now published in Journal of Mathematical Physics*, October 2015. arXiv:1510.04682.

# References III


[GEW16] Kenneth Goodenough, David Elkouss, and Stephanie Wehner. Assessing the performance of quantum repeaters for all phase-insensitive Gaussian bosonic channels. *New Journal of Physics*, 18(6):063005, June 2016. arXiv:1511.08710.

[HHH+08a] Karol Horodecki, Michał Horodecki, Paweł Horodecki, Debbie Leung, and Jonathan Oppenheim. Quantum key distribution based on private states: Unconditional security over untrusted channels with zero quantum capacity. *IEEE Transactions on Information Theory*, 54(6):2604–2620, June 2008. arXiv:quant-ph/0608195.

[HHH+08b] Karol Horodecki, Michał Horodecki, Paweł Horodecki, Debbie Leung, and Jonathan Oppenheim. Unconditional privacy over channels which cannot convey quantum information. *Physical Review Letters*, 100(11):110502, March 2008. arXiv:quant-ph/0702077.

[HHHO05] Karol Horodecki, Michał Horodecki, Paweł Horodecki, and Jonathan Oppenheim. Secure key from bound entanglement. *Physical Review Letters*, 94(16):160502, April 2005. arXiv:quant-ph/0309110.

# References IV


[HHHO09] Karol Horodecki, Michal Horodecki, Pawel Horodecki, and Jonathan Oppenheim. General paradigm for distilling classical key from quantum states. *IEEE Transactions on Information Theory*, 55(4):1898–1929, April 2009. arXiv:quant-ph/0506189.

[HW01] Alexander S. Holevo and Reinhard F. Werner. Evaluating capacities of bosonic Gaussian channels. *Physical Review A*, 63(3):032312, February 2001. arXiv:quant-ph/9912067.

[Li14] Ke Li. Second order asymptotics for quantum hypothesis testing. *Annals of Statistics*, 42(1):171–189, February 2014. arXiv:1208.1400.

[MLDS+13] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum Rényi entropies: a new generalization and some properties. *Journal of Mathematical Physics*, 54(12):122203, December 2013. arXiv:1306.3142.

[PLOB15] Stefano Pirandola, Riccardo Laurenza, Carlo Ottaviani, and Leonardo Banchi. Fundamental limits of repeaterless quantum communications. 2015. arXiv:1510.08863v6.

# References V


[PPV10] Yury Polyanskiy, H. Vincent Poor, and Sergio Verdú. Channel coding rate in the finite blocklength regime. *IEEE Transactions on Information Theory*, 56(5):2307–2359, May 2010.

[TBR15] Marco Tomamichel, Mario Berta, and Joseph M. Renes. Quantum coding with finite resources. *now published in Nature Communications*, April 2015. arXiv:1504.04617.

[TGW14] Masahiro Takeoka, Saikat Guha, and Mark M. Wilde. Fundamental rate-loss tradeoff for optical quantum key distribution. *Nature Communications*, 5:5235, October 2014. arXiv:1504.06390.

[TH13] Marco Tomamichel and Masahito Hayashi. A hierarchy of information quantities for finite block length analysis of quantum tasks. *IEEE Transactions on Information Theory*, 59(11):7693–7710, November 2013. arXiv:1208.1478.

# References VI


[TSW16] Masahiro Takeoka, Kaushik P. Seshadreesan, and Mark M. Wilde. Unconstrained distillation capacities of a pure-loss bosonic broadcast channel. *Proceedings of the 2016 IEEE International Symposium on Information Theory*, pages 2484–2488, July 2016. Barcelona, Spain. arXiv:1601.05563.

[TT15] Marco Tomamichel and Vincent Y. F. Tan. Second-order asymptotics for the classical capacity of image-additive quantum channels. *Communications in Mathematical Physics*, 338(1):103–137, August 2015. arXiv:1308.6503.

[TWW14] Marco Tomamichel, Mark M. Wilde, and Andreas Winter. Strong converse rates for quantum communication. June 2014. arXiv:1406.2946.

[WTB16] Mark M. Wilde, Marco Tomamichel, and Mario Berta. Converse bounds for private communication over quantum channels. February 2016. arXiv:1602.08898.

[WTLB16] Mark M. Wilde, Marco Tomamichel, Seth Lloyd, and Mario Berta. Gaussian hypothesis testing and quantum illumination. August 2016. arXiv:1608.06991.

# References VII

[WWY14] Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. *Communications in Mathematical Physics*, 331(2):593–622, October 2014. arXiv:1306.1586.