Towards new upper bounds for QMA(k)

Martin Schwarz

Dahlem Center for Complex Quantum Systems
Freie Universität Berlin, D-14195 Berlin, Germany

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What is known so far...

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$$\mathsf{QMA} \subseteq \mathsf{QMA}(2) \subseteq \mathsf{NEXP}$$

- only trivial bounds known so far for the general case
- under some assumptions, non-trivial bounds are known:
  - [BCY11], [SW15] show quasi-polynomial time algorithms (in d) for h_{sep} assuming $\|\cdot\|_F$-norm bounds on measurement operators
  - both based on geometric methods (ε-nets, SDP hierarchies)
  - don't exploit circuit structure of verifier at all → new angle?

[BCY11] Brandao, Christandl, Yard. Proc. ACM STOC (2011)
[SW15] Shi, Wu. Theor. Comp. Sci. 598 (2015) 51–63

Chailloux and Sattath´s result

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- Search for a QMA(2)-complete problem [CS11]
- Obvious candidate: **Local Hamiltonian (LH)**, but with *separable witnesses state*, called **Separable Local Hamiltonian (SLH)**
- Turns out, **Separable Local Hamiltonian** is in fact QMA-complete!
- Nevertheless, [CS11] show **Separable <u>Sparse</u> Hamiltonian (SSH)** is to be QMA(2)-complete
- **Sparse** vs. **Local** - where is the big difference?

[CS11] Chailloux, Sattath. IEEE Conf. Comp. Compl. (2011)

Harrow-Montanaro normal form

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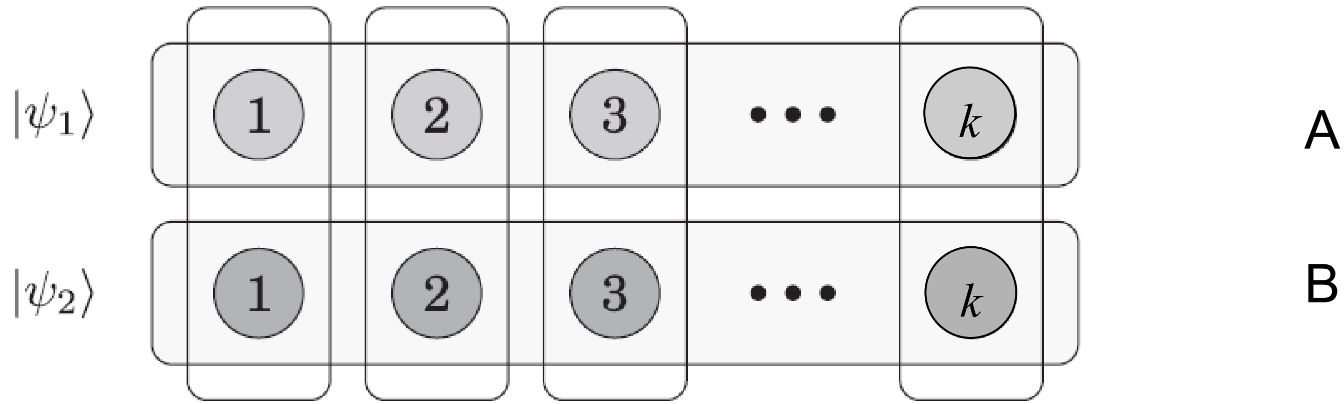

- every QMA(k) verifier can be reduced to a QMA(2)-verifier [HM13], such that:
  - the two witness states in the A:B subsystems are identical copies
  - only k-many $poly(n)$-qubit SWAP tests are performed across A:B
  - the original verifier circuit is only executed on either A or B
  - in the YES case, measurement is *separable* operator across A:B!

[HM13] Harrow, Montanaro. Journal of the ACM, Vol. 60, No. 1, Article 3

Separable Sparse Hamiltonian

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$$H = H_{in} + H_{prop} + H_{out} \qquad H_{prop} = \sum_{t=1}^{T} H_t$$

$$H_t = -\frac{1}{2}|t\rangle\langle t-1| \otimes U_t - \frac{1}{2}|t-1\rangle\langle t| \otimes U_t^\dagger + \frac{1}{2}(|t\rangle\langle t| + |t-1\rangle\langle t-1|) \otimes I$$

- [HM13] verifier circuit mapped to a Local Hamiltonian yields:
  - k groups of *poly(n)*-many local terms encoding CSWAP unitaries across the A:B boundary at the beginning of the computation
  - *poly(n)*-many terms acting only on one subsystem, encoding the verifier

- Problem: history state of this Hamiltonian is NOT separable!

  → single-qubit CSWAPs at different times create entanglement

- Solution: CSWAP all qubits of each group in a “single time step“!

  → witness state now separable, but k terms now sparse, but non-local

My starting point

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$$H_{\mathrm{CSWAP}} = -\frac{1}{2}\overbrace{\mathrm{SWAP}_{1,\ell+1}\otimes\mathrm{SWAP}_{2,\ell+2}\otimes\cdots\otimes\mathrm{SWAP}_{\ell,2\ell}}^{\text{swap terms } S_i}\otimes\overbrace{|1\rangle\langle 1|}^{\text{control } C}\otimes\overbrace{(|t\rangle\langle t-1|+|t-1\rangle\langle t|)}^{\text{time propagation } T}$$

- Can we simulate such terms (more) locally?
- Is there a “parallel Feynman-Kitaev construction“ that “executes“ parallel unitaries in the same “time step“ in the history state?
- Idea: use perturbation theory gadgets to reduce locality!
  - used before to reduce 5-local to 2-local QMA-complete Hamiltonians
  - exponential cost in Hamiltonian norm
- A reduction in promise gap might be acceptable, though...

Perturbation theory review

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- Start with a simple, unperturbed Hamiltonian with a large spectral gap Δ between ground state and excited states, acting on ancilla w:

$$H = \Delta |1\rangle\langle 1|_w$$

- H acts trivially on other qubits → degenerate GS

- Adding a perturbation V with $||V||<\Delta/2$, can (under certain conditions) approximate the spectrum of a desired effective Hamiltonian H_{eff} in the low-energy spectrum (E $<\Delta/2$) of

$$\tilde{H} = H + V$$

Perturbation theory review

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Theorem 10 (Eigenvalue-Approximating Perturbation Theorem [OT08, Theorem 7]). *Let* $||V|| \leq \Delta/2$ *where* Δ *is the spectral gap of* H *and* $\lambda(H) = 0$. *Let* $\tilde{H}|_{<\Delta/2}$ *be the restriction of* $\tilde{H} = H + V$ *to the space of eigenstates with eigenvalues less than* $\Delta/2$. *Let there be an effective Hamiltonian* H_{eff} *with* $\text{Spec}(H_{\text{eff}}) \subseteq [a, b]$. *If the self-energy* $\Sigma_-(z)$ *for all* $z \in [a - \varepsilon, b + \varepsilon]$ *where* $a < b < \Delta/2 - \varepsilon$ *for some* $\varepsilon > 0$, *has the property that*

$$||\Sigma_-(z) - H_{\text{eff}}|| \leq \varepsilon, \tag{5}$$

then each eigenvalue $\tilde{\lambda}_j$ *of* $\tilde{H}|_{<\Delta/2}$ *is* ε*-close to the* j*th eigenvalue of* H_{eff}. *In particular*

$$|\lambda(H_{\text{eff}}) - \lambda(\tilde{H})| \leq \varepsilon. \tag{6}$$

- Generic perturbation theorem for any choice of V and H_{eff}, where the
- *Self-energy expansion* $\Sigma_-(z)$ of $\tilde{H}$ converges to H_{eff}
- If gadget V is more local than H_{eff} → effective *locality reduction!* [OT08]

- <u>Alternatives:</u> *Bloch expansion* or *Schrieffer-Wolff transformation*
- <u>Observation:</u> No statement at all about eigen<u>vectors</u>!

[OT08] Oliveira, Terhal. Quant. Inf, Comp. Vol. 8, No. 10 (2008)

Perturbation theory review

Theorem 11 (Norm-Approximating Perturbation Theorem [OT08, Theorem A.1]). *Given is a Hamiltonian H such that no eigenvalues of H lie between $\lambda_- = \lambda_* - \Delta/2$ and $\lambda_+ = \lambda_* + \Delta/2$. Let $\tilde{H} = H + V$ where $||V|| \leq \Delta/2$. Let there be an effective Hamiltonian H_{eff} with $\text{Spec}(H_{\text{eff}}) \subseteq [a, b]$, $a < b$. We assume that $H_{\text{eff}} = \Pi_- H_{\text{eff}} \Pi_-$. Let D_r be a disk of radius r in the complex plane centered around $z_0 = \frac{b+a}{2}$. Let r be such that $b + \varepsilon < z_0 + r < \lambda_*$. Let $w_{\text{eff}} = \frac{b-a}{2}$. Assume that for all $z \in D_r$ we have $\|\Sigma_-(z) - H_{\text{eff}}\| \leq \varepsilon$. Then*

$$\|\tilde{H}_{<\lambda_*} - H_{\text{eff}}\| \leq \frac{3(||H_{\text{eff}}|| + \varepsilon)\|V\|}{\lambda_+ - ||H_{\text{eff}}|| - \varepsilon} + \frac{r(r + z_0)\varepsilon}{(r - w_{\text{eff}})(r - w_{\text{eff}} - \varepsilon)}. \tag{7}$$

- But even more is true: approximation in *operator norm!*
- Implies that orginal witness space is close to perturbed one
- Original 5-LH to 2-LH reductions only cared about eigenvalues
  - structure of eigenvectors of no relevance – only promise gap important

[OT08] Oliveira, Terhal. Quant. Inf, Comp. Vol. 8, No. 10 (2008)

Subdivision Gadget

Concrete choice of V to reduce locality, also works in parallel [OT08]:

$$H = \Delta|1\rangle\langle 1|_w\,, \quad V = H'_{\mathrm{else}} + \sqrt{\Delta/2}\,(-A + B) \otimes X_w$$

$$\tilde{H} = H + V$$

$(l+1)$ -local

$$H_{\mathrm{target}} = H'_{\mathrm{else}} + A \otimes B \longleftarrow 2l \text{ -local}$$

$$H_{\mathrm{eff}} \;=\; H_{\mathrm{target}} \otimes |0..0\rangle\langle 0..0|$$

Lemma: *If $\Delta > poly(n,k)/\varepsilon^2$, then* $\left\|\tilde{H}|_{<\Delta/2} - H_{\mathrm{target}} \otimes |0\ldots 0\rangle\langle 0\ldots 0|\right\| \le \varepsilon$

[OT08] Oliveira, Terhal. Quant. Inf, Comp. Vol. 8, No. 10 (2008)

Subdivision Gadget

- Breaking l-local down to 3-local terms [CBBK15]:

iteration (tree depth) i

$S_1 S_2 S_3 S_4 | S_5 C T$

$i = 1$

$S_1 S_2 S_3 | S_4 X_{u_1}$ $\quad$ $X_{u_1} S_5 | C T$

$i = 2$

$S_1 S_2 | S_3 X_{u_2}$ $\quad$ $X_{u_2} S_4 X_{u_1}$ $\quad$ $X_{u_1} S_5 X_{u_3}$ $\quad$ $X_{u_3} C T$

$i = 3$

$S_1 S_2 X_{u_4}$ $\quad$ $X_{u_4} S_3 X_{u_2}$

- Exponential scaling of gap Δ in l [BDLT08]:

$$\Delta_{i+1} = \frac{\mathrm{poly}(n,k)}{\varepsilon^2} \Delta_i^3 \qquad \Delta = \left(\frac{\mathrm{poly}(n,k)}{\varepsilon^2} \right)^{3^{\log(\ell)}} = \left(\frac{\mathrm{poly}(n,k)}{\varepsilon^2} \right)^{\mathrm{poly}(\ell)} = 2^{O(\mathrm{poly}(\ell) \log(nk/\varepsilon))}$$

[BDLT08] Bravyi, DiVincenzo, Loss, Terhal. PRL 101(7):070503 (2008)
[CBBK15] Cao, Babbush, Biamonte, Kais. PRA 91(1):012315 (2015)

QMA(2): **SSH** to **SLH** reduction

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- Applying the subdivision gadget to our **SSH** instance, we can break down the k non-local but sparse terms to 3-local ones, yielding **SLH**

- Definition of **Local Hamiltonian** requires poly-bounded norm of H, but this reduction increases it exponentially

→ Solution: rescale **SLH** instance dividing by Δ

- Operator norm of H now $O(1)$
- Soundess/completness divided by Δ → *promise gap* divided by Δ

→ SSH to SLH conversion at the cost of exp. small promise gap! (...)

Deciding **SLH** using ε-nets

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Corollary 6 ([SW15, Problem 3, Corollary 6]). *Take the expression* $Q = \sum_{i=1}^{r} H_i$ *of any* l*-local Hamiltonian over* $A_1 \otimes \cdots \otimes A_k$ *(each* A_i *is of dimension* $d = 2^n$*) such that* $\|H_i\|_{\mathrm{op}} \leq w$ *for each* i *as input. Assuming* $k, l = O(1)$*, the quantity*

$$OptSep(Q) = \min \langle Q, \rho \rangle \text{ subject to } \rho \in SepD(A_1 \otimes \cdots \otimes A_k) \tag{1}$$

where $SepD(A_1 \otimes \cdots \otimes A_k)$ *is the set of separable density operators over the space* $A_1 \otimes \cdots \otimes A_k$*, can be approximated to precision* δ *in*

$$\mathsf{DTIME}(\exp(O(\log^{O(1)}(d)(\log\log(d) + \log(w/\delta)))) \times \mathrm{poly}(d, w, 1/\delta)), \tag{2}$$

which is quasi-polynomial in $d, w, 1/\delta$*. If* n *is considered as the input size and* $w/\delta = O(\mathrm{poly}(n))$*, then OptSep(Q) can be approximated to precision* δ *in* PSPACE.

→ yields an **EXP** algorithm for $d=2^n$ and $\delta=1/2^{poly(n)}$, thus “$\mathrm{QMA}(2) \subseteq$ EXP“

→ if there weren‘t a problem lurking...

[SW15] Shi, Wu. Theoretical Computer Science 598 (2015) 51–63

Soundness and completeness

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- **The Problem:** the perturbation theory requires projection onto the low-energy subspace!

$$\left\| \tilde{H}|_{<\Delta/2} - H_{\mathrm{target}} \otimes |0\ldots0\rangle\langle 0\ldots0| \right\| \leq \varepsilon$$

- Projected Hamiltonian: no longer local (self-defeating perturbation)
- Projected separable states: close in energy, but no longer separable
- Unprojected separable states: **overlap with high-energy subspace**

- Soundness OK: high energy states stay high energy
- Completeness NOK: original witness has too much energy

Analyzing the issue

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Geometric picture: Schrieffer-Wolff transformation [BDL11, BLDT08]

$$H_0 = \Delta|1\rangle\langle 1|_u, \;\; V = \sqrt{\Delta J/2}\, X_u \otimes (-A+B) + V_{\text{extra}} \qquad S = -i\sqrt{\frac{J}{2\Delta}}\, Y_u \otimes (-A+B)$$

$$e^S H e^{-S} = \left(H + [S,H] + \frac{1}{2}[S,[S,H]]\right) + O(J^{3/2}\Delta^{-1/2}) = \begin{bmatrix} H_{\text{target}} & 0 \\ 0 & \Delta I + O(J) \end{bmatrix} + O(J^{\frac{3}{2}}\Delta^{-\frac{1}{2}})$$

→ H_{target} is reproduced in a slightly rotated basis (up to some error)
→ overlap with excited subspace is penalized by Δ

How much overlap with excited space *(1-P)* of H can an ground state of H_0 have?

Lemma: Let P_0 be projector on eigenvectors of H_0. Let P be projector on (<$\Delta/2$)–eigenvectors of H. Then, $\|P - P_0\| \leq \frac{2J\,\|V\|}{\Delta} = \sqrt{\frac{2J^3}{\Delta}}$

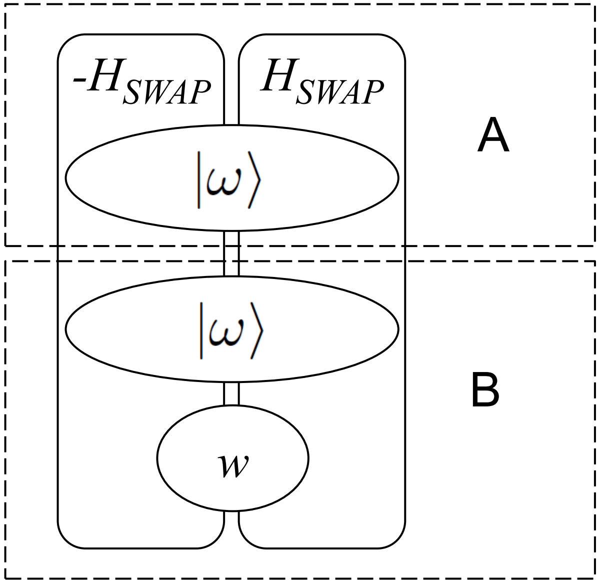
→ Thus, upper bound on energy: $\langle\psi|\langle 0|H|\psi\rangle|0\rangle \leq O(\sqrt{\Delta})$

$O(\frac{1}{\sqrt{\Delta}})$ overlap with $(1-P)$

$O(\Delta)$ energy penalty on $(1-P)$

[BDL11] Bravyi, DiVincenzo, Loss. Ann. Phys. Vol. 326, No. 10 (2011)
[BDLT08] Bravyi, DiVincenzo, Loss, Terhal. PRL 101(7):070503 (2008)

Example for violation



$$\tilde{H} = H + V \quad H = \Delta|1\rangle\langle 1|_w \quad \Delta = 100$$

$$V = \mathbb{1} - \underbrace{\sqrt{\frac{\Delta}{2}}\mathrm{SWAP}_1 \otimes X_w}_{H_{\mathrm{SWAP}}} + \underbrace{\sqrt{\frac{\Delta}{2}}\mathrm{SWAP}_2 \otimes X_w}_{H_{\mathrm{SWAP}}}$$

$$H_{\mathrm{eff}} = -\mathrm{SWAP}_1 \otimes \mathrm{SWAP}_2 \otimes |0\rangle\langle 0|_w$$

$$|\psi\rangle = |\omega\rangle \otimes |\omega\rangle \otimes |0\rangle \quad |\omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- $P|\psi\rangle$ necessarily has small superposition on ancilla
- superposition on ancilla+B doesn‘t really help much

$$\langle\psi|H_{\mathrm{eff}}|\psi\rangle = -1$$

$$\langle\psi|P\tilde{H}P|\psi\rangle = -0.94$$

$$\langle\psi|\tilde{H}|\psi\rangle = +1$$

Alternative constructions

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- Find new “parallel“ Feynman-Kitaev Hamiltonians, e.g.
  - implement one time step with k ancillas as parallel clocks
  - each cock ancilla couples to one of the CSWAP gates (→ local)
  - couple k ancillas with n.n. -ZZ terms to enforce synchronized tick
  - add small local -X terms to single out $|+\rangle = |00...0\rangle + |11...1\rangle$

  → Ising model (ZZ+X terms): has analytical solution!
  - similar to Jordan-Farhi [JF08] direct k-local to 2-local gadget

  → -X needed to single out $|+\rangle = |00...0\rangle + |11...1\rangle$ as ground state

  → otherwise, degenerate ground space, includes <u>no</u> time propagation

  → but X acts like perturbation: non-zero amplitudes in GS, i.e. partially flipped clocks (non-separable), original witness energy too high

[JF08] Jordan, Farhi. PRA 77, 062329 (2008)

QMA(2) in PSPACE?

- Assume the reduction to $\textbf{SLH}_{\textbf{gap}}$ worked, thus $\mathrm{QMA}(2) \subseteq \mathrm{EXP}$
- [CS11] also show that **SLH** (poly-gap) is QMA-complete
  - Reduces to *Consistency of Density Matrices* (CDM) problem
  - Witness consists of classical density matrices and quantum consistency proof
  - CDM uses QMA+ verifier which requires $O(poly(1/\varepsilon))$ copies of witness state
- If we could show that $\textbf{SLH}_{\textbf{gap}}$ is in $\mathrm{QMA}_{\mathrm{exp}}$, then we can use $\mathrm{QMA}_{\mathrm{exp}} = \mathrm{PSPACE}$ [FL16] to put $\mathrm{QMA}(2) \subseteq \mathrm{PSPACE}$
- What about a very direct reduction of $\mathrm{QMA}(2)$ to $\mathrm{QMA}_{\mathrm{exp}}$?
  - E.g. receive one copy and “guess“ the second one with exp. small chance?

[FL16] Fefferman, Lin. arXiv:1601.01975 (2016)
[CS11] Chailloux, Sattath. IEEE Conf. Comp. Compl. (2011)

Conclusion

SSH to **SLH** reduction (with small gap):

- ✗ Subdivision gadget seems to be self-defeating for QMA(2)
- Does not rule out alternative gadget designs, maybe very specific one
- Does not rule out alternative “parallel“ Feynman-Kitaev constructions, that maintain locality by construction

Solving **SLH** with small gap:

- ✓ in EXP using ε-net algorithm of Shi and Wu
- might even be in PSPACE by reduction to $\mathrm{QMA}_{\mathrm{exp}} = \mathrm{PSPACE}$

Thank you!