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The complexity of the Separable Hamiltonian Problem

QuICS workshop on QMA(2), 2016

Main results

Two variants of the LOCAL-HAMILTONIAN problem. We show:

- SEPARABLE SPARSE HAMILTONIAN is QMA(2)-Complete.
- SEPARABLE LOCAL HAMILTONIAN is in QMA!

Motivation

- We have an entire toolbox for working with Hamiltonians. Might be a way to prove upperbounds on QMA(2)(next talk?).
- This difference between the sparse and local case is surprising.

Quantum Merlin-Arthur (QMA)

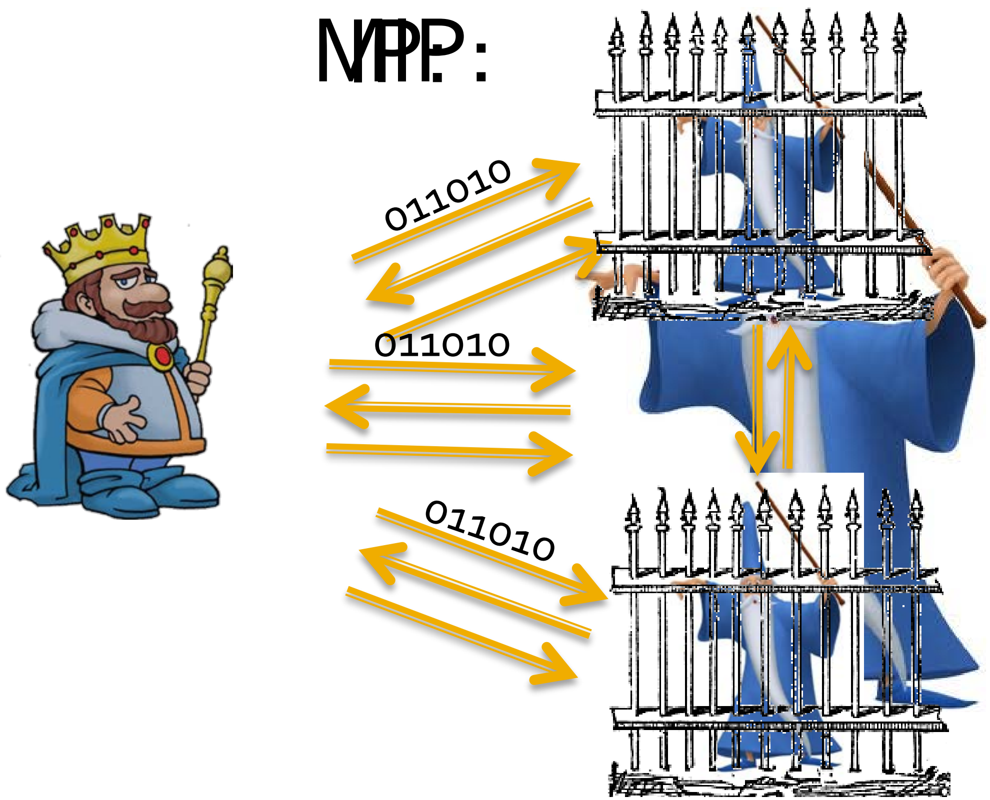
- Merlin is all powerful, but can be malicious, Arthur is skeptical, and limited to BQP.
- A problem $L \in QMA$ if:
  - $x \in L \Rightarrow \exists|\psi\rangle$ that Arthur accepts w.h.p.
  - $x \notin L \Rightarrow \forall|\psi\rangle$ Arthur rejects w.h.p.

$|\psi\rangle$

Restricting the prover

- Restricting the prover can increase the size of a complexity class:

MIP: IP:



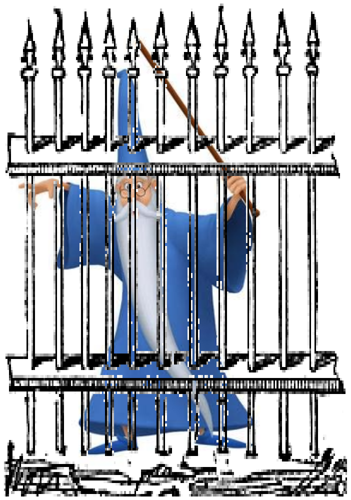
= RE = PSPACE = NEXP

QMA(2)

- Merlin A and Merlin B are all powerful, do not share entanglement.
- Arthur is skeptical, and limited to BQP.
- A problem $L \in QMA(2)$ if:
- $x \in L \Rightarrow \exists |\psi_1\rangle \otimes |\psi_2\rangle$ that Arthur accepts w.h.p.
- $x \notin L \Rightarrow \forall |\psi_1\rangle \otimes |\psi_2\rangle$ Arthur rejects w.h.p.

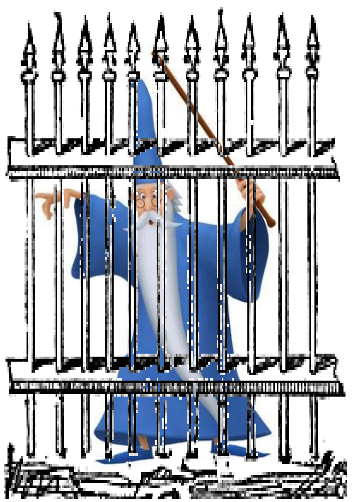
QMA(2)

$|\psi_A\rangle$



$\otimes$

$|\psi_B\rangle$



“The power of unentanglement”

- There are short proofs for NP-Complete problems in QMA(2) [BT’07,Beigi’10,LNN’11,ABD+`09].
- PURE N-REPRESENTABILITY $\in$ QMA(2) [LCV’07].
- QMA(k) = QMA(2) [HM’10].
- QMA $\subseteq$PSPACE, while the best upper-bound we had was QMA(2) $\subseteq$ NEXP [KM’01].
- Later today: attempt towards QMA(2) $\subseteq$ EXP [Schwarz’15], building on top of this result.

The LOCAL HAMILTONIAN problem

- The canonical QMA-Complete problem.
- Problem: Given $H = \sum_i H_i$, where each H_i acts non trivially on k qubits, decide whether there exists a state with energy below a or all states have energy above b?
- Witness: the state with energy below a.
- Verification: estimation of the energy.

The Separable Hamiltonian problem

- Problem: Given $H = \sum_i H_i$, and a partition of the qubits, decide whether there exists a separable state with energy at most a or all separable states have energies above b?
- A natural candidate for a QMA(2)-Complete problem.
- The witness: the separable state with energy below a.

Local Hamiltonian is QMA-Complete

- Given a quantum circuit Q, a history state is a state of the form:
  $\left|\eta_\psi\right\rangle \sim \sum_{t=0}^{T} |t\rangle \otimes |\psi_t\rangle$,
  where $|\psi_t\rangle \equiv U_t \ldots U_1 |\psi\rangle$.
- Kitaev's Hamiltonian gives an energetic penalty to:
  - states which are not history states.
  - history states are penalized for Pr(Q rejects $|\psi\rangle$)

Adapting Kitaev's Hamiltonian for QMA(2) – naïve approach

What happens if we use Kitaev's Hamiltonian for a QMA(2) circuit, and allow only separable witnesses?

Problems:

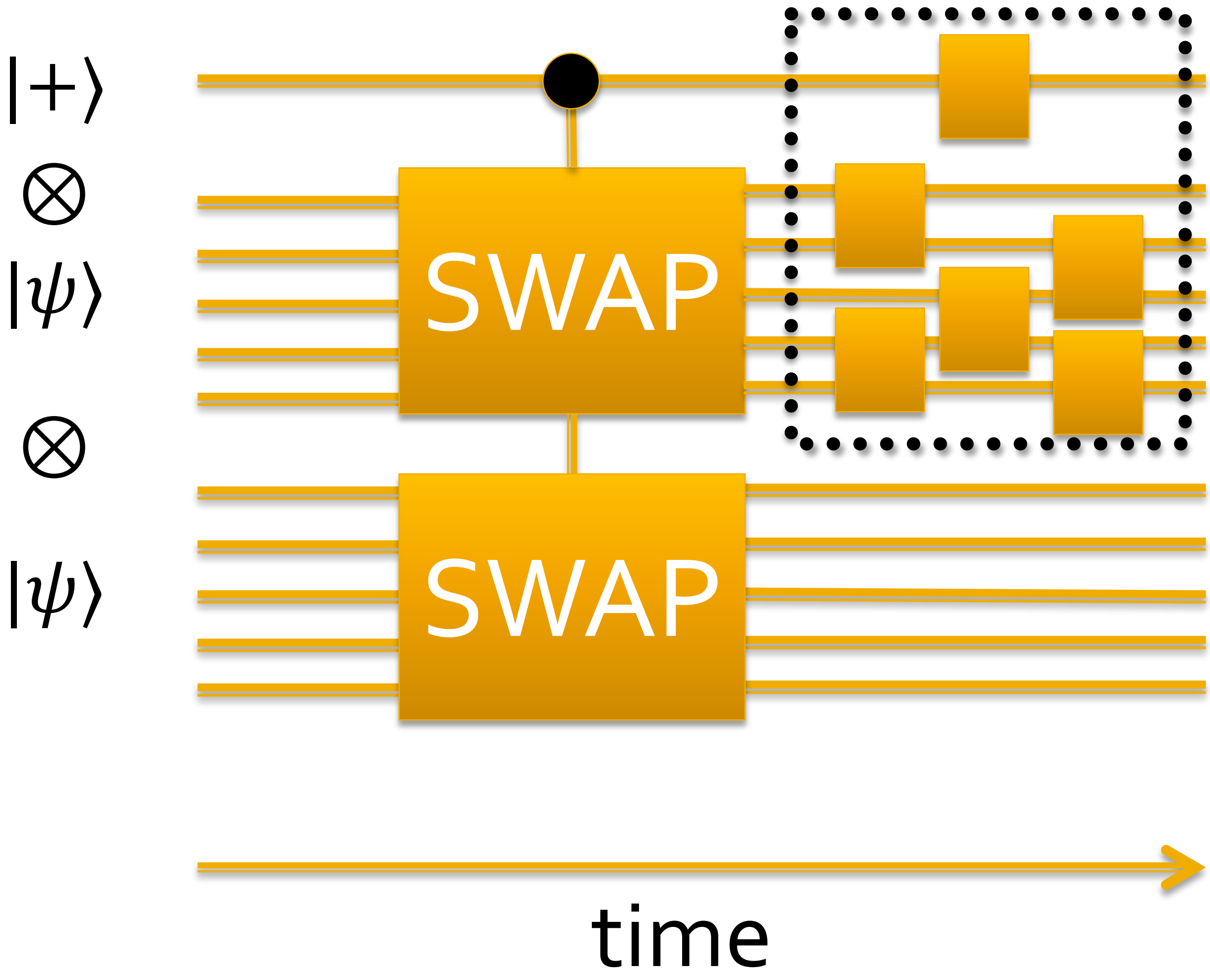
- Even if $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then $|\psi_t\rangle$ is typically entangled.
- Even if $\forall t \ |\psi_t\rangle$ **is** separable , $|\eta_\psi\rangle$ is typically entangled.

For this approach to work, one part must be fixed & non-entangled during the entire computation.

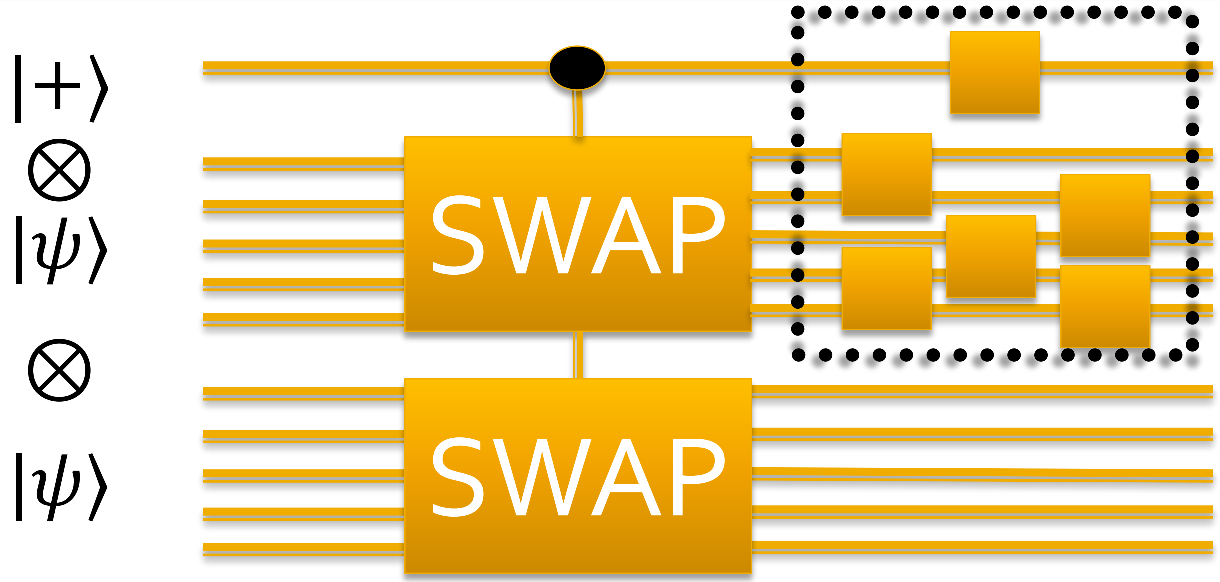
Adapting Kitaev's Hamiltonian for QMA(2)

- We need to assume that one part is fixed & non-entangled during the computation.
- Aram Harrow and Ashley Montanaro have shown exactly this!
- Thm: Every QMA(k) verification circuit can be transformed to a QMA(2) verification circuit with the following form:

Harrow-Montanaro's construction



Adapting Kitaev's Hamiltonian for QMA(2)

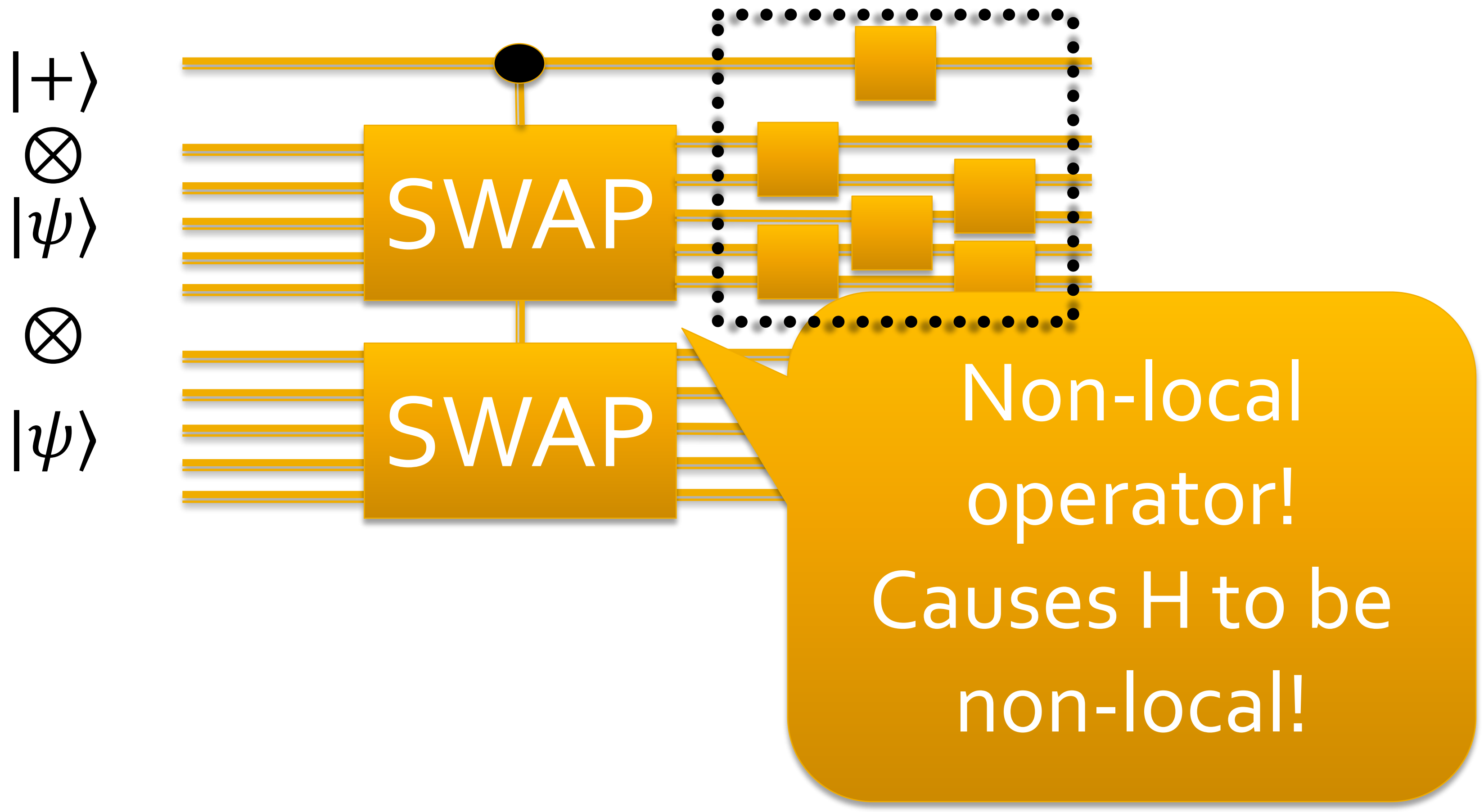


The history state is a tensor product state:

$$|\eta\rangle = \left(\sum_{t=0}^{T} |t\rangle \otimes U_t \dots U_1 |\psi\rangle \right) \otimes |\psi\rangle$$

Adapting Kitaev's Hamiltonian for QMA(2)

- There is a flaw in the argument:



Adapting Kitaev's Hamiltonian for QMA(2)

- Control-Swap over multiple qubits is sparse:
- C-SWAP= $\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & & 1 & \\ & & & & & 1 & & \\ & & & & & & & 1 \end{pmatrix}$.
- The energy of sparse-Hamiltonians can be estimated, and therefore

Sparse vs Local

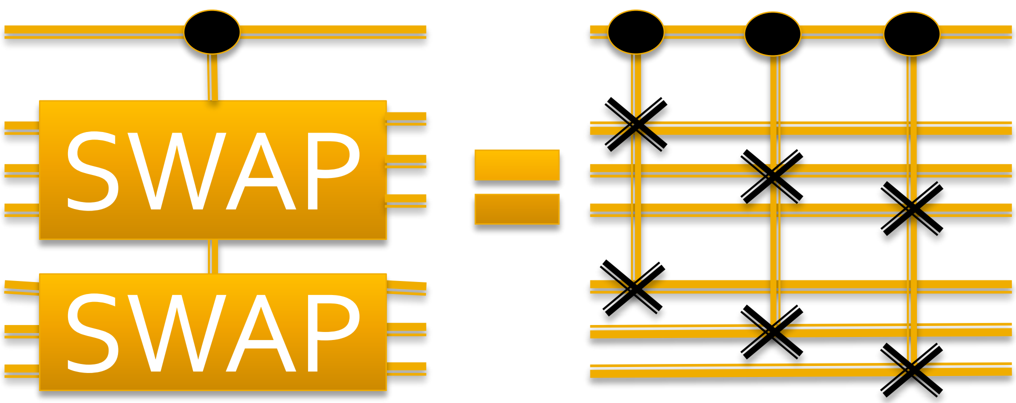
- Sparse Hamiltonians and Local Hamiltonians share the following properties:
  - Every Local Hamiltonian is sparse
  - Both variants can be simulated efficiently
  - The energy of a Sparse Hamiltonian (just like a local Hamiltonian) can be estimated, and therefore Sparse Hamiltonian is QMA-Complete [Aharonv-Ta Shma'03]
  - Simulation with exponentially small error to both [Berry et al.'13]
  - We were used to think that whatever holds for local Hams. holds for sparse Hams.

Adapting Kitaev's Hamiltonian for QMA(2)

- SPARSE HAMILTONIAN∈QMA [Aharonv-Ta Shma'03].
- SEPARABLE SPARSE-HAMILTONIAN∈QMA(2), for the same reason.
- The instance that we constructed is local, except one term which is sparse.
- **Theorem**: SEPARABLE SPARSE HAMILTONIAN is QMA(2)-Complete.

Adapting Kitaev's Hamiltonian for QMA(2)

- **<u>Theorem</u>**: SEPARABLE SPARSE HAMILTONIAN is QMA(2)-Complete.
- Can we further prove: SEPARABLE LOCAL HAMILTONIAN is QMA(2)-Complete?



- If we use the local implementation of C-SWAP, the history state becomes entangled.
- Seems like a technicality. Is it?

Separable Local Hamiltonian

- **<u>Thm 2:</u>** Separable Local Hamiltonian $\in$ QMA.
- Proof sketch: The prover sends the classical description of the reduced density matrices for ρ_A, and ρ_B.
- The verifier can calculate the energy classically:

$$Tr(H\rho_A \otimes \rho_B) = \sum_i Tr(H_i \rho_i)$$

- The prover also sends a quantum proof that these two states (with these reduced density matrices) exist.

Consistency of Local Density Matrices

- Consistensy of Local Density Matrices Problem (CLDM):
  Input: density matrices $\sigma_1, \dots, \sigma_m$ over k qubits and sets $A_1, \dots, A_m \subset \{1, \dots, n\}$.
- Output: yes, if there exists an n-qubit state ρ which is consistent: $\forall i \leq m,\ \rho^{A_i} = \sigma_i$. No, otherwise.
- Theorem[Liu`06]: CLDM ∈ QMA.

Verification procedure for Separable Local Hamiltonian

- The prover sends:
  a) Classical part, containing the reduced density matrices of ρ_A, and ρ_B.
  b) A quantum proof for the fact that ρ_A exists, and similarly, that ρ_B exists.
- The verifier:
  a) classically verifies that the energy is below the threshold a, assuming that the state is $\rho_A \otimes \rho_B$.
  b) verifies that there exists such a state ρ_A and ρ_B, using the CLDM protocol.

Verification procedure for Separable Local Hamiltonian

- When H_i involves both parts, the verifier assumes that the state is $\rho_A \otimes \rho_B$. It exists because the prover already proved the existence of both ρ_A and ρ_B.

Summary

- Known results: LOCAL HAMILTONIAN & SPARSE HAMILTONIAN are QMA-Complete.
- A "reasonable" guess would be that their Separable version are either QMA(2)-Complete, or QMA-Complete, but it turns out this is wrong.
- SEPARABLE SPARSE HAMILTONIAN is QMA(2)-Complete.
- SEPARABLE LOCAL HAMILTONIAN is QMA-Complete.

Open problem 1: Pure N-representability

- Similar to CLDM, but with the additional requirement that the state is pure (i.e. not a mixed state).
- In QMA(2): verifier receives 2 copies, and estimates the purity using the swap test:

$$\Pr(\sigma \otimes \tau \text{ passes the swap test}) = Tr(\sigma\tau) \leq \sqrt{Tr(\sigma^2)}.$$

- Is pure N-representability QMA(2)-Complete?

Open problem 2: Refereed Hamiltonian Problem

- L in QRG(1) if:
  - $x \in L \Rightarrow \exists\rho\forall\sigma\, \Pr(V\ accepts\ \rho \otimes \sigma) \geq 2/3$
  - $x \notin L \Rightarrow \exists\sigma\forall\rho\, \Pr(V\ accepts\ \rho \otimes \sigma) \leq 1/3$
- Refereed Hamiltonian Problem:
  - $H \in L \Rightarrow \exists\rho\forall\sigma\, Tr(H\ \rho \otimes \sigma) \leq a$
  - $H \notin L \Rightarrow \exists\sigma\forall\rho\, Tr(H\ \rho \otimes \sigma) \geq b$

where b-a < 1/poly(n)

- Is Refereed local / sparse Hamiltonian QRG(1)-Complete?
- Known results:

$$\mathrm{P}^{QMA} \subseteq P^{QRG(1)} = QRG(1) \subseteq QRG(2) = PSPACE \subseteq QRG = EXP$$